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**LINEAR ALGEBRA APPLIED TO PHYSICS**

**DETERMINING SMALL VIBRATIONS IN CONSERVATIVE  
ELASTIC SYSTEMS**

**LT COL SHIRLEY J. HAMILTON**

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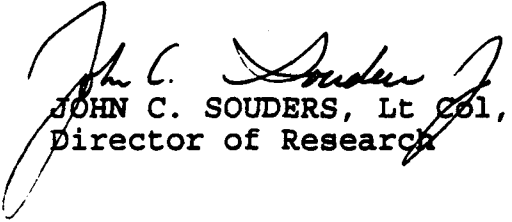
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**Linear Algebra Applied to Physics**

**Determining Small Vibrations in Conservative Elastic Systems**

by

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## Linear Algebra Applied to Physics

### Determining Small Vibrations in Conservative Elastic Systems

**Linear Algebra Prerequisites:** Being able use eigenvalues and eigenvector to diagonalize a symmetric matrix.

**Prerequisite Knowledge in Physics:** None.

**Other Prerequisite Knowledge:** A background in solving basic differential equations would be helpful. However, Appendix A contains this basic information.

## Section 1 Introduction

In this study we will look at small vibrations. In particular, the small vibrations which we will study are in a system with an **equilibrium configuration** which is a position where the system remains at rest. An example of a system in its equilibrium configuration is the simple pendulum as seen in Figure 1.1. The simple pendulum consists of a ball attached to a taut wire, anchored above, which can swing in the vertical plane. The weight of the wire is negligible compared to the weight of the ball.

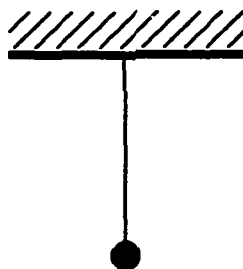


Figure 1.1

We say that a system has a **stable equilibrium configuration** if after a small displacement, the system tends to return to its equilibrium configuration. There are different types of equilibrium depending on the nature of the system. We are interested in the type of equilibrium found in an **elastic system**. This is a system which has the following two characteristics: 1) the system has a stable equilibrium configuration and 2) a small displacement from equilibrium creates forces which tend to restore the system to its

stable equilibrium configuration. A displacement from equilibrium is called **strain** and the force which restores the system to equilibrium is called **stress**. Thus, stress is a function of strain. The simple pendulum in Figure 1.1 is also an example of an elastic system in its stable equilibrium configuration.

The total energy in an elastic system is composed of two types of energy, kinetic and potential. We will begin by considering the intuitive definitions of these terms and then discuss their formulas. **Kinetic energy** is the energy a body possesses because it is in motion. Before we can write the formula for kinetic energy, we must be able to describe the system mathematically. In any system there is a minimum number of coordinates that are required to fully describe the configuration of the system. In general, the number of coordinates is equal to the number of "particles" in the system times the dimension of the system. In the case of the simple pendulum, the ball is the only particle in the system. The dimension of the system is one, because the position of the ball can be described using the angle made by the pendulum compared to the position of the pendulum in its equilibrium configuration, as seen in Figure 1.1. Therefore, the number of coordinates needed to describe the simple pendulum is one. The velocity of the system can also be written in terms of the coordinates which describe the configuration of the system. To be able to do this, we must specialize our notation. If  $n$  coordinates ( $x_1, x_2, \dots, x_n$ ) are required to describe the system, then each  $x_i$  represents a Cartesian coordinate of one of the particles in



the system. For example, if we have two particles moving in the xy-plane, which has dimension two, we will need four coordinates to describe the system. The four coordinates  $(x_1, x_2, x_3, x_4)$  represent the Cartesian coordinates of the particles in the system; that is,  $x_1$  and  $x_2$  represent x- and y-coordinates of the first particle, and  $x_3$  and  $x_4$  represent the x- and y-coordinates of the second particle.

From this we see that the velocity of the system can be expressed in terms of the velocity of each coordinate. The velocity vector for a system with n coordinates can be written in terms of its velocity components

$$\left( \frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt} \right)$$

The kinetic energy of the system is equal to the sum of one half the square of each velocity component times the mass of the particle which the coordinate describes. If we let T represent kinetic energy and  $m_i$  the mass of the particle which is described using the Cartesian coordinate  $x_i$ , then our formula becomes

$$T = \frac{1}{2} \sum_{i=1}^n m_i \left( \frac{dx_i}{dt} \right)^2$$

To have a **conservative system**, there must exist a function whose partial derivative with respect to any coordinate, say  $x_i$ , is equal to the negative value of the force in the direction represented by that coordinate. This function is called the **potential energy function**. We can describe the relationship between this function and the forces in the system by the equation

$$\frac{\partial}{\partial x_i}(\text{potential energy function}) = -(\text{force in the } x_i \text{ direction})$$

From now on, we will assume that we are always in a conservative system. In addition, if the potential energy function is not time dependent  $\left( \frac{\partial}{\partial t}(\text{potential energy function}) = 0 \right)$ , then in our conservative elastic system the total energy of the system is constant and is the sum of the kinetic and potential energies. Also, when the strain of the system is zero (the system is in its equilibrium configuration, so  $x_i = 0$  for all  $i$ ), then the partial derivative of the potential energy function with respect to any variable must equal zero. This statement can be interpreted in the following two ways: 1) in the equilibrium configuration the potential energy function is at a minimum and 2) the restoring forces are equal in magnitude and of opposite sign to the forces that created the displacement. This statement also tells us that the potential energy function can not contain linear terms which have nonzero constant coefficients in any of the  $x_i$ . To see why this is true, let us

assume the potential energy function contains a nonzero linear term  $cx_i$  ( $c$  is a nonzero constant). Then take the partial derivative of it with respect to  $x_i$ . Setting  $x_i$  equal to zero, we find the nonzero

constant  $c$  is equal to zero, which is a contradiction. Therefore, we conclude that  $c$  must be zero and the potential energy function does not contain a nonzero linear term  $cx_i$ . Also, it does not matter if the

potential energy function has a constant term or not, because when we differentiate the function with respect to any  $x_i$  ( $i=1, 2, \dots, n$ ), the

constant becomes zero. Thus, if we write the potential energy

function in its Taylor series expansion, the non-constant part starts with quadratic and terms of higher powers (which may also contain a constant term). When we differentiate the potential energy function with respect to  $x_i$  (for  $i=1, 2, \dots, n$ ), we obtain a linear

combination of the variables  $x_1, x_2, \dots, x_n$  plus higher order or

mixed terms (for example  $x_1x_n$  or  $x_1x_2x_n$ ). If we ignore the higher

order terms, then the linear part which remains gives us the specific relationship of stress to strain, which is known as Hooke's Law. In

general, Hooke's Law states that "stress is a linear transformation operating on strain." Intuitively, we would say, restoring forces are linearly proportional to the displacement of the mass from

equilibrium. If the non-constant potential energy function starts with a power greater than two, it is possible to use an approximation

to find the relationship between stress and strain in which the still higher power terms in the partial derivative of the potential energy function have been ignored. However, this is no longer a linear function.

We will consider two approaches to the formulation of a differential equation which models a system. The first approach is developed using Newton's second and third laws of motion, which are stated below for convenience.

**Second Law** - The mass of the body times the acceleration of the body is equal to the force acting on the body.

**Third Law** For every action there is an opposite and equal reaction.

From these laws we derive the differential equation which models a conservative elastic system

$$\text{mass} \times \frac{d^2(\text{displacement})}{dt^2} = \text{restoring force}$$

or

$$(1.1) \quad \text{mass} \times \frac{d^2(\text{strain})}{dt^2} = \text{stress}.$$

In the second approach, instead of using the direct application of Newton's laws, we will consider a method developed by Joseph Lagrange, a French mathematician. This very elegant and sophisticated method can be applied to systems which are more general than the ones we are considering here. Since our system is conservative and elastic, the energy is constant and equal to the sum of the kinetic and potential energies. If we let  $V$  represent potential energy and  $E$  represent the total energy in an  $n$  coordinate system, then we have

$$E = T + V = \frac{1}{2} \sum_{i=1}^n m_i \left( \frac{dx_i}{dt} \right)^2 + V$$

To make it easier to express this differential equation, we will introduce a type of notation you may not have used before. The derivative  $\frac{dx}{dt}$  will be written as  $\dot{x}$ , where the single dot above the variable  $x$  indicates that one derivative of  $x$ , with respect to time, has been taken. This idea can be extended so that  $\ddot{x}$  indicates that two derivatives of  $x$  with respect to time have been taken. This notation is used to rewrite the equation for total energy.

$$(1.2) \quad E = T + V = \frac{1}{2} \sum_{i=1}^n m_i \dot{x}_i^2 + V$$

We wish to derive the equations of motion which can be used to model conservative elastic systems. Our first step is to find the

partial derivative of Equation (1.2) with respect to each of the coordinates. Since the procedure is the same when taking the partial derivative with respect to each coordinate, we will only find the partial derivative of the function for total energy (a constant) with respect to the  $x_i$  th coordinate. We obtain

$$0 = \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i}$$

or

$$(1.3) \quad -\frac{\partial V}{\partial x_i} = \frac{\partial T}{\partial x_i}$$

Since the mass of each particle is known, the partial derivative of the kinetic energy with respect to the coordinate  $x_i$  is

$$\frac{\partial T}{\partial x_i} = \frac{1}{2} m_i \left[ 2 \dot{x}_i \frac{\partial \dot{x}_i}{\partial x_i} \right] = m_i \frac{\partial x_i}{\partial t} \frac{\partial \dot{x}_i}{\partial x_i} = m_i \frac{\partial \dot{x}_i}{\partial t} = m_i \ddot{x}_i$$

Substituting this into Equation (1.3), we obtain the restoring force of the  $x_i$  coordinate.

$$(1.4) \quad -\frac{\partial V}{\partial x_i} = m_i \ddot{x}_i$$

Since the kinetic energy is expressed in terms of  $\dot{x}_i$ , we can find

$$\frac{\partial T}{\partial \dot{x}_i}.$$

$$\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2} m_i \left[ 2 \dot{x}_i \right] = m_i \dot{x}_i$$

We now differentiate this equation with respect to time to obtain

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_i} \right] = m_i \frac{d\dot{x}_i}{dt} = m_i \ddot{x}_i$$

We see that the right side of this equation is the same as the right side of Equation (1.4). Equating the two, we obtain the equation of motion for the  $x_i$ th coordinate.

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_i} \right] = - \frac{\partial V}{\partial x_i}$$

Therefore, the equations of motion which model our conservative elastic system with  $n$  coordinates are

$$(1.5) \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = - \frac{\partial V}{\partial x_1}, \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = - \frac{\partial V}{\partial x_2}, \quad \dots, \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_n} \right] = - \frac{\partial V}{\partial x_n}.$$

We will model the simple pendulum of Figure 1.1 using both the method which applies Newton's laws directly and the equations

of motion formulated by Lagrange. Since we are only interested in small vibrations of a system, let us discuss the conditions under which the vibrations in the pendulum system remain small. In a conservative elastic system the total energy of the system is constant and is the sum of the kinetic and potential energies. The potential energy of the pendulum system is determined by the displacement of the ball from its equilibrium position. Imagine the pendulum in Figure 1.1 being placed very close to its equilibrium position and released as in Figure 1.2.

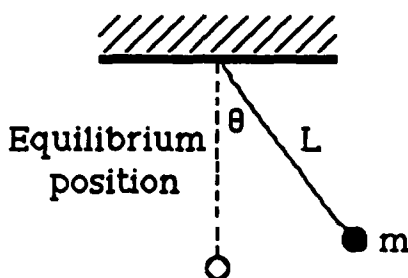


Figure 1.2

Since the potential energy is small to start with (the displacement from equilibrium is small) and we are in a conservative system, we know that it will remain small. Because the displacement stays small, the angle  $\theta$  will always be small. Thus, the vibrations of this system can only be small vibrations.

Let us model the simple pendulum system using the method which applies Newton's laws directly. To keep this example simple we will only consider the positive region which is to the right of the equilibrium configuration in Figure 1.3(a). Let  $L$  represent the



length of the pendulum,  $m$  be the mass of the ball at the end of the pendulum,  $\theta$  the angle the pendulum makes with respect to the equilibrium configuration, and  $s$  represent the length of the arc the ball travels. The force pulling the ball down is the mass of the ball times the acceleration due to gravity  $g$ .

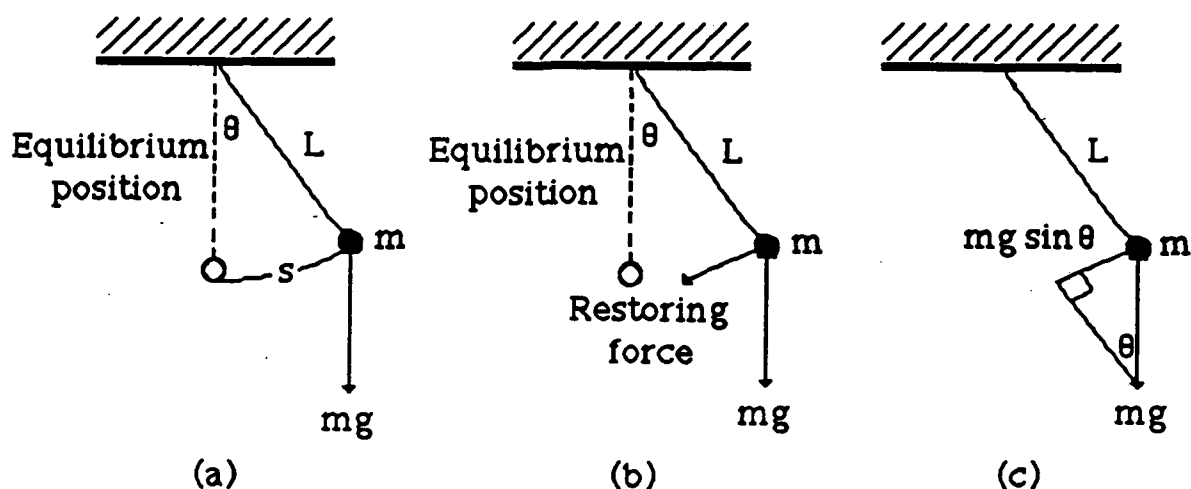


Figure 1.3

We will apply Newton's laws of motion to model the pendulum system by using Equation (1.1). Thus, we need to determine the stress and the strain of the system. Since stress or restoring force is the force trying to return the ball to its equilibrium configuration, we must resolve the force on the ball ( $mg$ ) into its component forces. Figure 1.3(b) shows the restoring force is the component of force on the ball along the arc length. Since  $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ , the magnitude of the restoring force is  $mg \sin \theta$  as seen in Figure 1.3(c). We will need a minus sign to indicate that the restoring force is

opposite in direction to the force which originally moved the ball from its equilibrium configuration. Thus, the restoring force or stress is equal to  $-mg \sin \theta$ . The strain is the displacement of the ball from the equilibrium position. This distance is the arc length  $s$ , which can also be described using the equation  $s=L\theta$ . Substituting these values for stress and strain into equation (1.1) gives

$$m \frac{d^2}{dt^2} [L \theta] = -mg \sin \theta$$

Taking the second derivative of  $L\theta$  with respect to time, this equation becomes

$$m L \ddot{\theta} = -mg \sin \theta$$

Simplifying and moving all terms to the left side of the equation, we get the second order differential equation that models our conservative elastic system.

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0$$

We now model the simple pendulum system using the equations of motion formulated by Lagrange. Since  $\theta$  is the only coordinate needed to describe the system, we will only need to use one of the equations of motion found in Equation (1.5).

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{\theta}} \right] = - \frac{\partial V}{\partial \theta}$$

Thus, we need to find both the kinetic energy and potential energy of the system. The kinetic energy is one half the mass of the ball times the square of the velocity. The velocity is the first derivative of the distance with respect to time.

$$\text{velocity} = \frac{d}{dt} [\text{distance}] = \frac{d}{dt} [L\theta] = L \frac{d\theta}{dt} = L\dot{\theta}$$

Therefore, the equation describing the kinetic energy becomes

$$T = \frac{1}{2} m [L\dot{\theta}]^2 = \frac{1}{2} m L^2 \dot{\theta}^2$$

Since potential energy is the energy needed to restore the system to equilibrium, it is equal to weight of the ball (mass of the ball times the acceleration due to gravity  $g$ ) times the height of the ball above the reference point. Since the ball is below the reference point,  $V$  is negative. Using the fact that  $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ , we determine the distance of the ball below the reference point to be  $L\cos\theta$  as seen in Figure 1.4.

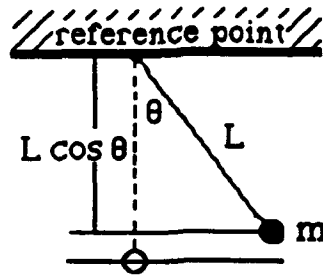


Figure 1.4

Thus, the potential energy is

$$V = -mgL \cos \theta.$$

Now, we substitute the appropriate partial derivatives of the kinetic and potential energy equations into the equation of motion. First, we find the left side of the equation of motion by differentiating the kinetic energy equation with respect to  $\dot{\theta}$ .

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} m L^2 (2\dot{\theta}) = m L^2 \dot{\theta}$$

Then differentiating with respect to time, we obtain

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{\theta}} \right] = m L^2 \ddot{\theta}$$

Second, the right side of the equation of motion is

$$-\frac{\partial V}{\partial \theta} = - \left[ -m g L \frac{\partial(\cos \theta)}{\partial \theta} \right] = - \left[ -m g L (-\sin \theta) \right] = -m g L \sin \theta$$

Equating the two sides, the equation of motion becomes

$$m L^2 \ddot{\theta} = -m g L \sin \theta$$

Simplifying and moving all terms to the left side of the equation, we obtain the second order differential equation that models our conservative elastic system. As expected, this is the same equation which we found by applying Newton's laws.

$$(1.6) \quad \ddot{\theta} + \frac{g}{L} \sin \theta = 0$$

Let us pause for a moment and discuss the relationship between the potential energy function and the component of force tangent to the path the ball travels (that is, in the direction of arc length). Recall that in a conservative system, the partial derivative of the potential energy function with respect to any direction, gives the negative of the force in that direction. That is,  $\frac{\partial V}{\partial s} = -F_s$  where  $F_s$  is the force in the direction of the arc length  $s$ . First, we need to write the potential energy function in terms of arc length  $s$ . We will use the fact that  $s = L\theta$ .

$$V = -mgL \cos\left(\frac{s}{L}\right)$$

We continue by differentiating this with respect to  $s$  to obtain

$$\frac{\partial V}{\partial s} = -m g L \left[ -\frac{1}{L} \sin\left(\frac{s}{L}\right) \right] = m g \sin \theta$$

Thus,  $F_s = -\frac{\partial V}{\partial s} = -m g \sin \theta$  is the restoring force, since the simple pendulum is described using only one coordinate. (Recall the discussion following Figure 1.3.)

So far we have found the second order differential equation which models the simple pendulum system using two different methods. Now, we are ready to consider how Equation (1.6) can be solved. Since this equation involves  $\sin \theta$ , we know it is a nonlinear differential equation. (See Appendix A for definition.) One technique used to find the exact solution (if that is possible) of a second order nonlinear differential equation is to first reduce it to a first order differential equation. Recall that, the equation of motion was derived from Equation (1.7). Since Equation (1.6) was found by using the equation of motion, we can use Equation (1.7) as our first order differential equation.

$$(1.7) \quad E = T + V = \frac{1}{2} m L^2 \left( \frac{d\theta}{dt} \right)^2 - m g L \cos \theta$$

We begin by solving for the squared derivative  $\left( \frac{d\theta}{dt} \right)^2$  in Equation (1.7).

$$(1.8) \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{2(E + m g L \cos\theta)}{m L^2}$$

Let us pause for a moment to assure ourselves that we could legitimately use Equation (1.7) as our first order differential equation. We will differentiate Equation (1.8) with respect to time and see that we get Equation (1.6).

$$2 \dot{\theta} \ddot{\theta} = \frac{2}{m L^2} \left[ 0 + m g L (-\dot{\theta} \sin\theta) \right]$$

When simplified, we see this is Equation (1.6).

$$\ddot{\theta} + \frac{g}{L} \sin\theta = 0$$

Taking the square root of both sides of Equation (1.8) and recalling that we are only considering the positive region to the right of the equilibrium configuration, we obtain

$$\frac{d\theta}{dt} = \sqrt{\frac{2(E + m g L \cos\theta)}{m L^2}}$$

One way to solve the differential equation above for  $t$ , is to isolate  $dt$  on one side of the equation and  $d\theta$  on the other. This technique is sometimes called separation of variables.

$$dt = \sqrt{\frac{mL^2}{2(E + mgL \cos \theta)}} d\theta$$

Integrating both sides of the above equation, we obtain

$$t = \int \sqrt{\frac{mL^2}{2(E + mgL \cos \theta)}} d\theta$$

This is an elliptic integral which can not be expressed in terms of elementary functions. Thus, to get any information about the solution to Equation (1.6), we must resort to numerical approximations or use the fact that we are dealing with a system involving small vibrations. We also note that the question of finding the inverse function  $\theta = \theta(t)$  of the function above is, at best, a numerical approximation problem and is not even useful in predicting values of  $\theta$  at a given time  $t$ , since we are dealing with small vibrations. As a point of interest, if we were not considering small vibrations, then the function  $t = t(\theta)$  and its inverse function  $\theta = \theta(t)$  would be the only tools with which we could obtain information about the system.

Using the fact that we are dealing only with small vibrations, we consider the factor  $\sin \theta$ , which makes Equation (1.6) a nonlinear differential equation. We can write  $\sin \theta$  as a Taylor series expanded about zero.

$$\sin \theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$



Since we are considering only small values of  $\theta$ , the terms in the expansion above which contain powers of  $\theta$  are, in practice, ignored (a very small number raised to a power greater than one becomes even smaller). Any time terms are ignored we expect a certain amount of error. To determine the exact amount of error would require the same type of calculation that it would take to solve the original equation. However, since we are considering only small vibrations, we are assured the amount of error will not affect the resulting solution. Thus, using the Taylor series expansion for  $\sin \theta$  we see that  $\sin \theta$  can be replaced by  $\theta$ , for small values of  $\theta$ . This substitution is only valid when we are dealing with small vibrations. Using this substitution, the second order differential equation (Equation (1.6) ) becomes

$$\ddot{\theta} + \frac{g}{L} \theta = 0$$

This is a second order linear differential equation whose solution is found using basic techniques from differential equations. (Basic solution techniques are found in Appendix A.) We obtain

$$\theta = c_1 \cos\left(\sqrt{\frac{g}{L}} t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}} t\right)$$

In summary, we have investigated two different ways to model a conservative elastic system. One method applies Newton's

second and third laws directly to the system to create the differential equation. The other method uses a technique developed by Lagrange, which was much easier to generalize and could be applied to many different types of systems. The derived equations of motion, greatly simplify the amount of work necessary to model a conservative elastic system. From these techniques, we found the second order nonlinear differential equation that models the simple pendulum. Since we considered only small vibrations, we found that the equation could be represented by a second order linear differential equation which has an elementary solution.

## Section 2 Linear Spring-Mass Systems

In the last section we considered a system which only needed one coordinate to completely describe the system. We now look at higher dimensional systems, such as spring-mass systems in which more than one coordinate is required to specify the state of the system. A spring-mass system is a conservative elastic system with a stable equilibrium position occurring when all of the coordinates are set equal to zero. To become familiar with spring-mass systems, we will first consider the one dimensional case. Figure 2.1 shows the system in its equilibrium configuration (the spring is not being stretched or compressed) where  $m$  is the mass of the block and  $L$  is the natural length of the spring. We are considering the spring-mass system moving along a horizontal track rather than hanging

vertically so that we do not have the added complication of describing how gravity affects the system.

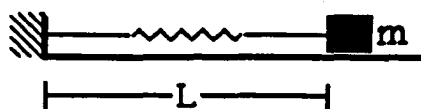


Figure 2.1

To determine the number of coordinates we need to describe this system, recall the formula given in Section 1. (The number of coordinates = (number of particles) times (dimension of the system).) The only particle in the system is the block and since the block is moving along a horizontal track, the dimension of the system is one. Thus, we need only one coordinate  $x$ , to describe the system. Imagine the block being moved to the right causing the spring to be stretched  $x$  units. This is shown in Figure 2.2.

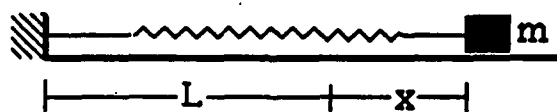


Figure 2.2

To describe the energy of this system we again need to find both the kinetic energy and potential energy. The kinetic energy is one half the mass of the block times the velocity squared. The equation describing the kinetic energy is

$$T = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 = \frac{1}{2} m \dot{x}^2$$

The energy stored in the spring or the potential energy of the spring is one half the spring constant times the square of the distance that the spring is stretched. From the laws of physics we know that the external force acting on the spring is proportional to the increase in length of the spring. We call the constant of proportionality that allows us to write this relationship as an equation, the spring constant or the stiffness of the spring and each spring has its own specific spring constant. If we let  $k$  represent the spring constant and  $x$  the displacement of the spring from equilibrium, then the equation for potential energy is

$$V = \frac{1}{2} k x^2$$

Since  $x$  is the only coordinate needed to describe the system, we will only need to use one of the equations of motion found in Equation (1.5).

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}} \right] = - \frac{\partial V}{\partial x}$$

To find the left side, we first differentiate the kinetic energy function with respect to  $\dot{x}$ .

$$\frac{\partial T}{\partial \dot{x}} = m \dot{x}$$

Now differentiate this equation with respect to time.

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}} \right] = m \ddot{x}$$

To find the right side of the equation of motion, we differentiate the potential energy function with respect to  $x$ .

$$-\frac{\partial V}{\partial x} = -k x$$

Equating these two, the equation of motion becomes

$$m\ddot{x} = -k x.$$

Simplifying and rearranging terms, the differential equation which models this system is

$$\ddot{x} + \frac{k}{m} x = 0$$

This is a second order linear differential equation whose solution is found using basic techniques from differential equations. (See Appendix A.) Note, the similarity between this differential equation and the one that models the simple pendulum.

$$x = c_1 \cos \left( \sqrt{\frac{k}{m}} t \right) + c_2 \sin \left( \sqrt{\frac{k}{m}} t \right)$$

To be able to model higher dimensional spring-mass systems, we need to study the theory which describes the energy of the system in general terms. In the one dimensional spring-mass system there was only one coordinate which we labeled as  $x$  and it was expressed in terms of time. The kinetic energy of the system was described using the first derivative of this coordinate with respect to time, while the potential energy was expressed in terms of the coordinate. If we are working with a higher dimensional system which has  $n$  coordinates, say  $x_1, x_2, \dots, x_n$ , then the kinetic energy will be described using the first derivative with respect to time of each of the coordinates and the potential energy will be expressed in terms of these  $n$  coordinates. In general we have

$$\text{Kinetic Energy} \quad T = \frac{1}{2} \sum_{i=1}^n m_i \left( \frac{dx_i}{dt} \right)^2 = \frac{1}{2} \sum_{i=1}^n m_i \dot{x}_i^2$$

$$\text{Potential Energy} \quad V = \sum_{i=1}^s V_i \quad \text{where } V_i \text{ is the potential energy of each spring and } s \text{ is the number of springs.}$$

Since stable equilibrium occurs when  $x_1 = x_2 = \dots = x_n = 0$ , we may assume that the energy of the system is at a minimum in stable equilibrium. This means the derivative with respect to any variable must be zero when that variable equals zero. Thus, if we have a

function which we wish to expand using its Taylor's series expansion, as we did with  $\sin \theta$  in Section 1, the expansion can not have a nonzero linear term. For if it did and we took the derivative of it, we would end up with a nonzero constant. Subsequently, when all variables are set equal to zero, the constant would remain, indicating that we do not have stable equilibrium, a contradiction. Therefore, the Taylor series expansion for the potential energy does not have linear terms. However, this expansion may have constant terms.

Let us return to the spring-mass system. Figure 2.3 shows a system in equilibrium with two blocks having the same mass  $m$  and three springs possessing the same length and spring constant. To determine the number of coordinates needed to describe this system, we need to recall the formula given in Section 1. (The number of coordinates = (number of particles) times (dimension of system)). The two particles in the system are the two blocks and since both blocks are moving along a horizontal track, the dimension of the system is one. Thus, we will need two coordinates,  $x_1$  and  $x_2$ , to describe the system.

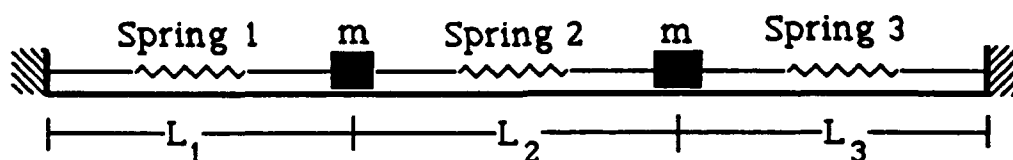


Figure 2.3

Imagine the two masses are moved to the right causing the first two springs to stretch by different amounts and causing the third spring to be compressed. This is depicted in Figure 2.4.

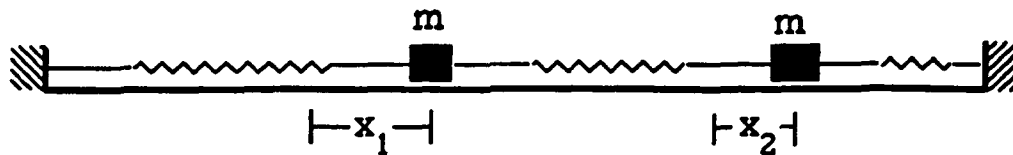


Figure 2.4

Now we determine the second order differential equation that models this system. Thus, we need to find the kinetic energy and the potential energy of the system. The equation below describes the kinetic energy of the system shown in Figure 2.4.

$$T = \frac{1}{2} \left[ m \left( \frac{dx_1}{dt} \right)^2 + m \left( \frac{dx_2}{dt} \right)^2 \right] = \frac{1}{2} m \left[ \dot{x}_1^2 + \dot{x}_2^2 \right]$$

The potential energy of the system is the sum of the potential energies of each spring. Spring 1 is stretched from its equilibrium position by the amount  $x_1$ , so the potential energy for spring 1 is

$V_1 = \frac{1}{2} k x_1^2$ . Spring 2 is stretched from its equilibrium position by the amount  $x_2 - x_1$ , so that  $V_2 = \frac{1}{2} k (x_2 - x_1)^2$  is the potential energy for spring 2. Spring 3 is compressed from its equilibrium position



by the amount  $x_2$ . Thus, the potential energy for spring 3 is

$V_3 = \frac{1}{2} k x_2^2$ . Therefore, the potential energy of the system is

$$V = \sum_{i=1}^3 V_i = \frac{1}{2} k \left[ x_1^2 + (x_2 - x_1)^2 + x_2^2 \right] = \frac{1}{2} k \left[ 2x_1^2 - 2x_1 x_2 + 2x_2^2 \right]$$

Since this system is described using two variables,  $x_1$  and  $x_2$ , our

two equations of motion are

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = - \frac{\partial V}{\partial x_1} \quad \text{and} \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = - \frac{\partial V}{\partial x_2}$$

First, we determine the left side of each equation of motion by differentiating the kinetic energy with respect to  $\dot{x}_1$  and  $\dot{x}_2$ .

$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{2} m [2 \dot{x}_1 + 0] = m \dot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2} m [0 + 2 \dot{x}_2] = m \dot{x}_2$$

Then, differentiating each of these equations with respect to time, we have

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = \frac{d}{dt} [m \dot{x}_1] = m \ddot{x}_1$$

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = \frac{d}{dt} [m \dot{x}_2] = m \ddot{x}_2$$

To determine the right side of each equation of motion, we differentiate the potential energy with respect to  $x_1$  and  $x_2$ .

$$-\frac{\partial V}{\partial x_1} = -\frac{1}{2}k[4x_1 - 2x_2 + 0] = k[-2x_1 + x_2]$$

$$-\frac{\partial V}{\partial x_2} = -\frac{1}{2}k[0 - 2x_1 + 4x_2] = k[x_1 - 2x_2]$$

Substituting this information into the equations of motion, we obtain

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = -\frac{\partial V}{\partial x_1}$$

which becomes

$$m \ddot{x}_1 = k[-2x_1 + x_2] \quad \text{or} \quad \ddot{x}_1 = \frac{k}{m}[-2x_1 + x_2]$$

and

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = -\frac{\partial V}{\partial x_2}$$

which becomes

$$m \ddot{x}_2 = k[x_1 - 2x_2] \quad \text{or} \quad \ddot{x}_2 = \frac{k}{m}[x_1 - 2x_2]$$

We now have a system of second order differential equations which can be written as the following matrix equation, where  $A$  is a symmetric matrix.

$$\vec{\ddot{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m} A \vec{X}$$

Since  $A$  is a symmetric matrix, all of its eigenvalues are real and  $A$  is diagonalizable. We begin the determination of the eigenvalues of the matrix  $A$  by

$$\det(A - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{pmatrix} = (-2 - \lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1).$$

If we set  $\det(A - \lambda I)$  equal to zero and solve for  $\lambda$ , we find the eigenvalues are  $\lambda = -3$  and  $\lambda = -1$ . Since  $A$  is diagonalizable there exists an invertible (orthogonal) matrix  $P$  such that  $P^{-1}AP = D$ . The matrix

$D$  is the diagonal matrix whose entries along the main diagonal consist of the eigenvalues of  $A$  and the columns of  $P$  are corresponding eigenvectors associated with these eigenvalues. To find  $P$  we need to find an eigenvector associated with  $\lambda = -3$  and one associated with  $\lambda = -1$ . For  $\lambda = -3$  we have the following matrix equation.

$$\begin{pmatrix} -2 + 3 & 1 \\ 1 & -2 + 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{so} \quad x_1 = -x_2$$

If we let  $x_1 = 1$ , then  $x_2 = -1$ , and it follows that an eigenvector

associated with the eigenvalue  $\lambda = -3$  is  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . To find an

eigenvector associated with  $\lambda = -1$ , we use the following matrix equation.

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ so } x_1 = x_2$$

If we let  $x_1 = 1$ , then  $x_2 = 1$ . Thus, an eigenvector associated with

the eigenvalue  $\lambda = -1$  is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . Therefore, these two eigenvectors are

the columns of the invertible matrix  $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ .

Recall that our goal is to solve the second order differential equation  $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$ . If we multiply both sides of this equation by  $P^{-1}$  and use the identity  $P P^{-1} = I_2$ , the  $2 \times 2$  identity matrix, we

obtain

$$(2.1) \quad P^{-1} \ddot{\vec{X}} = P^{-1} \frac{k}{m} A \vec{X} = \frac{k}{m} P^{-1} A (P P^{-1}) \vec{X} = \frac{k}{m} (P^{-1} A P) (P^{-1} \vec{X}).$$

We want to get Equation (2.1) into a simpler form to make it easier to solve. To do this we will let  $\vec{U} = P^{-1} \vec{X}$ . This matrix equation can

be easily differentiated with respect to time, since  $P^{-1}$  is a constant matrix. The first derivative with respect to time is  $\vec{\dot{U}} = P^{-1} \vec{\dot{X}}$ . Since Equation (2.1) is a second order differential equation, taking a second derivative with respect to time yields  $\vec{\ddot{U}} = P^{-1} \vec{\ddot{X}}$ . We introduce the vector variable  $\vec{U}$  into Equation (2.1) by substituting  $\vec{\dot{U}} = P^{-1} \vec{\dot{X}}$  into the left side of this equation. Then, if we substitute  $P^{-1}AP = D$  and  $\vec{U} = P^{-1} \vec{X}$  into the right side, Equation (2.1) becomes  $\vec{\ddot{U}} = \frac{k}{m} D \vec{U}$ . This system of second order differential equations is easier to solve than  $\vec{\ddot{X}} = \frac{k}{m} A \vec{X}$ .

### Exercise 2.1

Show why the system  $\vec{\ddot{U}} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{k}{m} D \vec{U}$  is

easier to solve than  $\vec{\ddot{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m} A \vec{X}$ .

Recall that we are trying to solve for the vector  $\vec{X}$ . Rearranging  $\vec{\dot{U}} = P^{-1} \vec{\dot{X}}$ , we get  $\vec{\dot{X}} = P \vec{\dot{U}}$ , which tells us that instead of finding  $\vec{\dot{X}}$  we need only find  $P \vec{\dot{U}}$ . Since we already know the matrix  $P$ , we must find the vector  $\vec{\dot{U}}$ . If we multiply both sides of the matrix equation  $\vec{\ddot{U}} = \frac{k}{m} D \vec{U}$  by the matrix  $P$ , we get

$P \vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$ . Rewriting the left side of  $P \vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$ , using the notation where  $P^{(i)}$  represents the  $i$ th column ( $i = 1$  and  $2$ ) of the matrix  $P$ , we obtain

$$P \vec{\ddot{U}} = (P^{(1)} P^{(2)}) \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = P^{(1)} \ddot{u}_1 + P^{(2)} \ddot{u}_2$$

Rewriting the right side of  $P \vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$ , we obtain

$$\frac{k}{m} P D \vec{U} = \frac{k}{m} [P^{(1)} \lambda_1 u_1 + P^{(2)} \lambda_2 u_2]$$

Thus  $P \vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$  can be written as

$$P^{(1)} \ddot{u}_1 + P^{(2)} \ddot{u}_2 = \frac{k}{m} [P^{(1)} \lambda_1 u_1 + P^{(2)} \lambda_2 u_2]$$

We can simplify this equation by multiplying through by  $\frac{k}{m}$ , gathering terms and moving all terms to the left side.

$$\left[ P^{(1)} \ddot{u}_1 - \frac{k}{m} P^{(1)} \lambda_1 u_1 \right] + \left[ P^{(2)} \ddot{u}_2 - \frac{k}{m} P^{(2)} \lambda_2 u_2 \right] = \vec{0}$$

Factoring out  $P^{(1)}$  and  $P^{(2)}$ , we have

$$\left[ \ddot{u}_1 - \frac{k}{m} \lambda_1 u_1 \right] P^{(1)} + \left[ \ddot{u}_2 - \frac{k}{m} \lambda_2 u_2 \right] P^{(2)} = \vec{0}$$

Since the columns of  $P$  are eigenvectors of  $A$  which correspond to distinct eigenvalues, we know they are linearly independent (in fact they are orthogonal). The equation produces a finite linear combination of linearly independent vectors which equals zero, thus the coefficients of  $P^{(1)}$  and  $P^{(2)}$  must be zero. If we set each of the coefficients in the equation above equal to zero, we obtain

$$\ddot{u}_1 - \frac{k}{m} \lambda_1 u_1 = 0 \quad \text{and} \quad \ddot{u}_2 - \frac{k}{m} \lambda_2 u_2 = 0$$

These are both second order linear differential equations which can be solved using basic techniques. (See Appendix A.) If we let

$r_i = -\frac{k}{m} \lambda_i$ , where  $i=1, 2$ , then these equations become

$$\ddot{u}_1 + r_1 u_1 = 0 \quad \text{and} \quad \ddot{u}_2 + r_2 u_2 = 0$$

Using the following formulas, we can solve for the vector  $\vec{U}$ . (Note: To determine whether  $r_i$  is zero, negative or positive, substitute  $\lambda_i$

into  $r_i = -\frac{k}{m} \lambda_i$ .)

If  $r_i = 0$ , then  $u_i = c_{i1}t + c_{i2}$

If  $r_i < 0$ , then  $u_i = c_{i1}e^{\sqrt{r_i}t} + c_{i2}e^{-\sqrt{r_i}t}$

If  $r_i > 0$ , then  $u_i = c_{i1}\cos(\sqrt{r_i}t) + c_{i2}\sin(\sqrt{r_i}t)$

The solution to the original system of differential equations

$\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$  is found by substituting the values for both the matrix  $P$  and the vector  $\vec{U}$  into the equation  $\vec{X} = P \vec{U}$ .

### Exercise 2.2

Using the above technique, solve the following system of differential

equations  $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$ . Where  $\vec{X} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$ ,  $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$  and

$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ . That is, find the two equations which describe  $x_1$  and  $x_2$ .

### Exercise 2.3

Given a horizontal spring-mass system similar to Figure 2.3 with  $n$  blocks, the following equations would express the kinetic and potential energies of the system.



Kinetic Energy

$$T = \frac{1}{2} m \sum_{i=1}^n \dot{x}_i^2$$

Potential Energy

$$\begin{aligned}
 V = \frac{1}{2} k \bigg[ & b_{11} x_1^2 + b_{22} x_2^2 + \dots + b_{nn} x_n^2 \\
 & + 2b_{12} x_1 x_2 + 2b_{13} x_1 x_3 + \dots + 2b_{1n} x_1 x_n \\
 & + 2b_{23} x_2 x_3 + 2b_{24} x_2 x_4 + \dots + 2b_{2n} x_2 x_n \\
 & \vdots \\
 & + 2b_{i, i+1} x_i x_{i+1} + \dots + 2b_{i, n} x_i x_n \\
 & \vdots \\
 & + 2b_{n-2, n-1} x_{n-2} x_{n-1} + 2b_{n-2, n} x_{n-2} x_n \\
 & + 2b_{n-1, n} x_{n-1} x_n \bigg]
 \end{aligned}$$

Use the equations of motion to find the system of differential equations which model the spring-mass system with  $n$  blocks. To solve this system, generalize the procedure used to solve the system of differential equations which model the spring-mass system with two blocks. (Hint: Some of the material that has been discussed can be used directly, while other portions will need some modifications.)

---

Another aspect of the spring-mass system that we want to consider is the oscillations of the system as a whole. From our work

above, we know the solution to the second order differential equation  $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$  can be found by using  $\vec{X} = P \vec{U}$ . In Exercise 2.2 we found

$$x_1 = u_1 + u_2 \quad \text{and} \quad x_2 = -u_1 + u_2.$$

This is a system of two linear equations which we can solve for  $u_1$  and  $u_2$ . Thus we have the equations

$$(2.2) \quad u_1 = \frac{x_1 - x_2}{2} \quad \text{and} \quad u_2 = \frac{x_1 + x_2}{2}$$

each of which gives a relationship between the variables  $x_1$  and  $x_2$ .

It is important to note that we could have found these equations directly from the matrix equation  $\vec{U} = P^{-1} \vec{X}$ , but this would involve finding  $P^{-1}$ . Using Figure 2.5, we can recall the configuration of

this spring-mass system. Since the springs were stretched by differing amounts, a different frequency (the number of vibrations per unit time) is associated with each of the variables  $x_1$  and  $x_2$ .

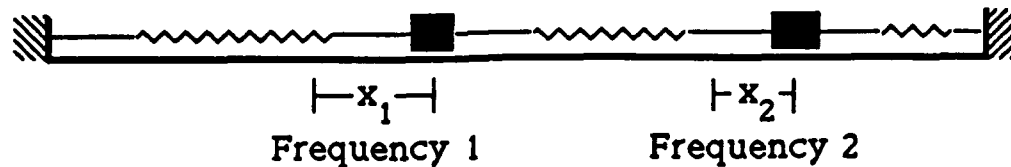


Figure 2.5

This spring-mass system has two separate modes in which it vibrates.

In the first mode  $u_1 = \frac{x_1 - x_2}{2}$  and  $u_2 = 0$ . Since  $\frac{x_1 - x_2}{2}$

represents the how the distance between the two blocks is changing, the first mode of vibration describes how the distance between the two blocks is changing. For instance if  $x_2$  is greater than  $x_1$ , then the change in distance between the blocks is smaller than the distance between the blocks when the spring-mass system is in equilibrium. However, if  $x_1$  is greater than  $x_2$ , then the change

in distance between the blocks is larger than the distance between the blocks when the spring-mass system is in equilibrium. Thus, the oscillation of the system in this mode is described by how the distance between the two blocks is changing which corresponds to the frequency associated with the second eigenvalue  $\lambda_2$ . To

visualize this, consider the series of "snapshots" of the spring-mass system in motion in Figure 2.6, where the banner is made of an elastic material and indicates the distance between the two blocks

in the system. When this system vibrates, we would see the banner contracting and stretching with a frequency associated with  $\lambda_2$ .

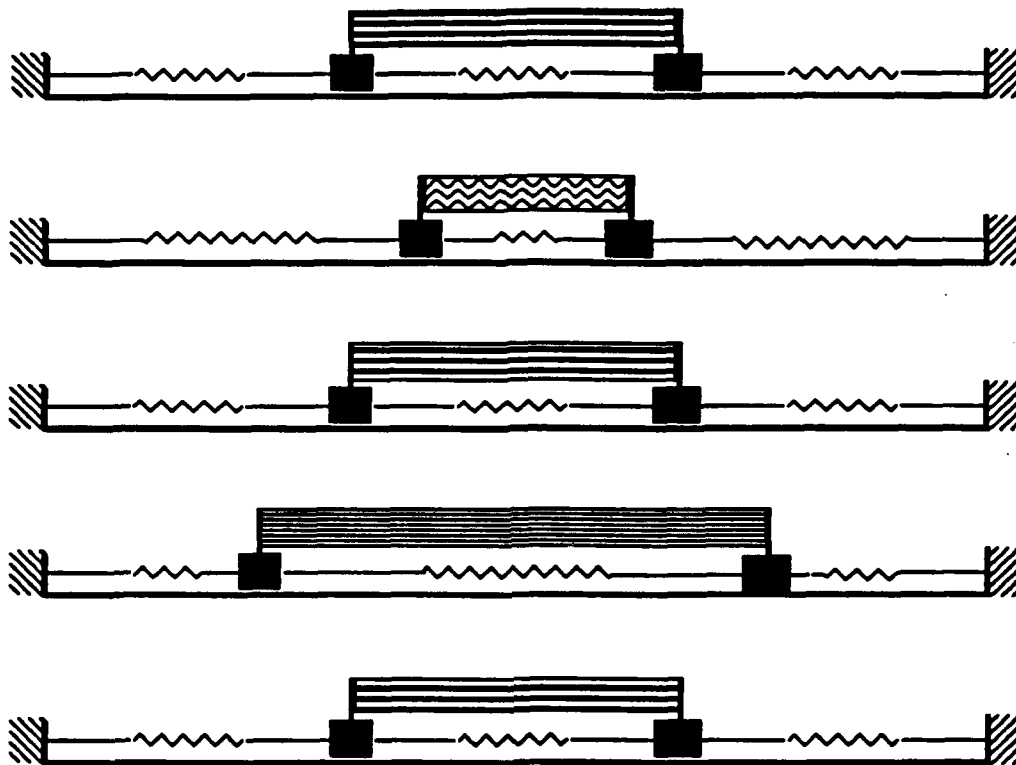


Figure 2.6

In the second mode  $u_1 = 0$  and  $u_2 = \frac{x_1 + x_2}{2}$ . Since  $\frac{x_1 + x_2}{2}$

represents how the center of mass of the system has changed, the second mode of vibrations describes the displacement of the center of gravity. Thus the oscillation of the system in this mode is where the center of mass of the system vibrates at the frequency associated with the first eigenvalue  $\lambda_1$ . To visualize this, consider the series of diagrams in Figure 2.7, where the flag indicates the

center of mass of the system. When this system vibrates, we see the flag moving back and forth with a frequency associated with  $\lambda_1$ .

This is indicated by the following series of "snapshots" of the spring-mass system in motion.

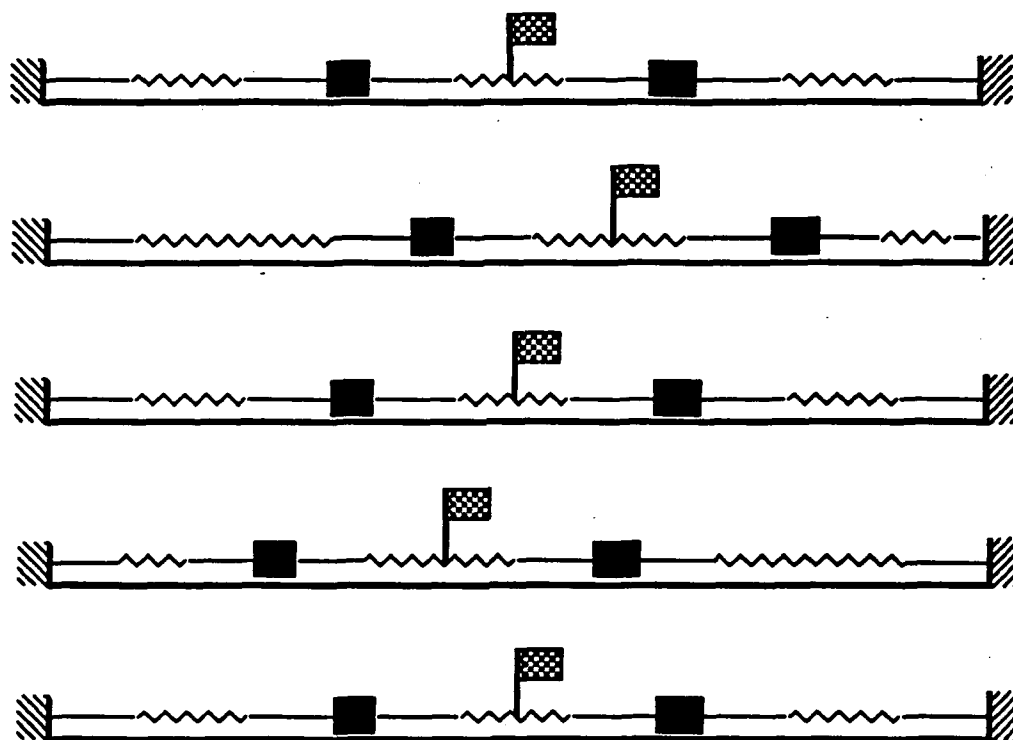


Figure 2.7

#### Exercise 2.4

Suppose the spring-mass system we have been studying was lying free in the  $xy$ -plane, that is, the ends of the springs are not anchored. Figure 2.8 can help us visualize this.

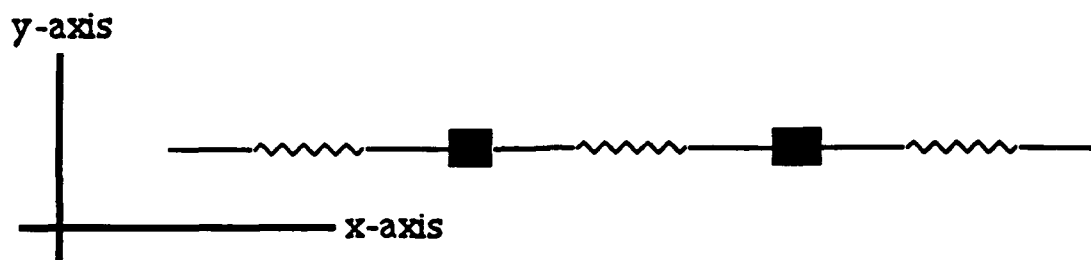


Figure 2.8

Using the information we have gained by studying the stationary spring-mass system, describe the motion (including the vibrations) that can occur. Note, there is no need to find the frequencies to complete this exercise. (Hint: consider other types of motion, besides vibrations.)

### Exercise 2.5

- Determine the system of differential equations that model the motions of the spring-mass system given in Figure 2.9.
- Solve the system of differential equations.
- Describe the possible configurations in which it vibrates.

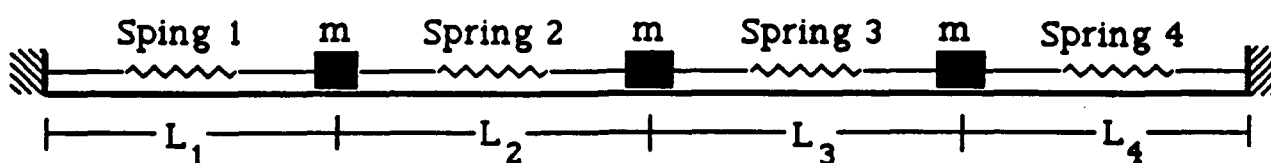


Figure 2.9

**Exercise 2.6**

Suppose the spring-mass system of Exercise 2.5 was lying free in the  $xy$ -plane, that is, the ends of the springs are not anchored.

Describe the motion (including vibrations) that can occur for this system. Compare these motions with the motions found in Exercise 2.4.

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**Section 3 A Closed Spring-Mass System**

In this section we will discuss how to mathematically model the spring-mass system in Figure 3.1 and determine the possible motions of the system. This system lies in the  $xy$ -plane with none of its blocks anchored. The mass of each of the three blocks is the same and is denoted by  $m$ .  $L$  is the length of each spring when the system is in its equilibrium configuration and  $k$  is the spring constant, which is the same for each spring.

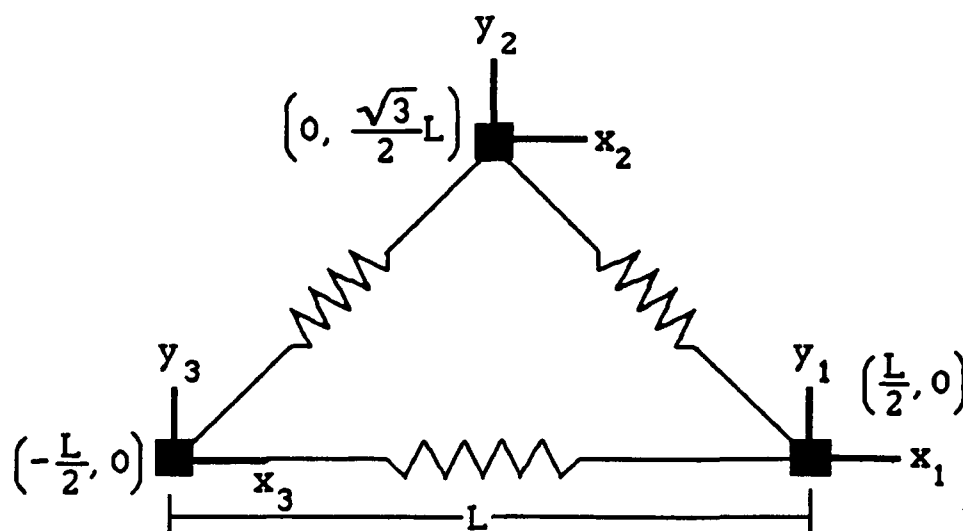


Figure 3.1

This system is in stable equilibrium when the displacements

$x_1 = y_1 = x_2 = y_2 = x_3 = y_3 = 0$ . To find the energy of the system in Figure

3.1 we need to find the kinetic and potential energies of the system. Recall that the kinetic energy of the system is one half the mass times the sum of the square of the first derivative of each of the six variables with respect to time. Thus, the kinetic energy is

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2)$$

Finding the potential energy requires more work. Since the potential energy of the system is the sum of the potential energies of the springs, we first need to find the potential energy of each spring. We will consider each side of the triangle individually.

The first side that we look at is given in Figure 3.2.



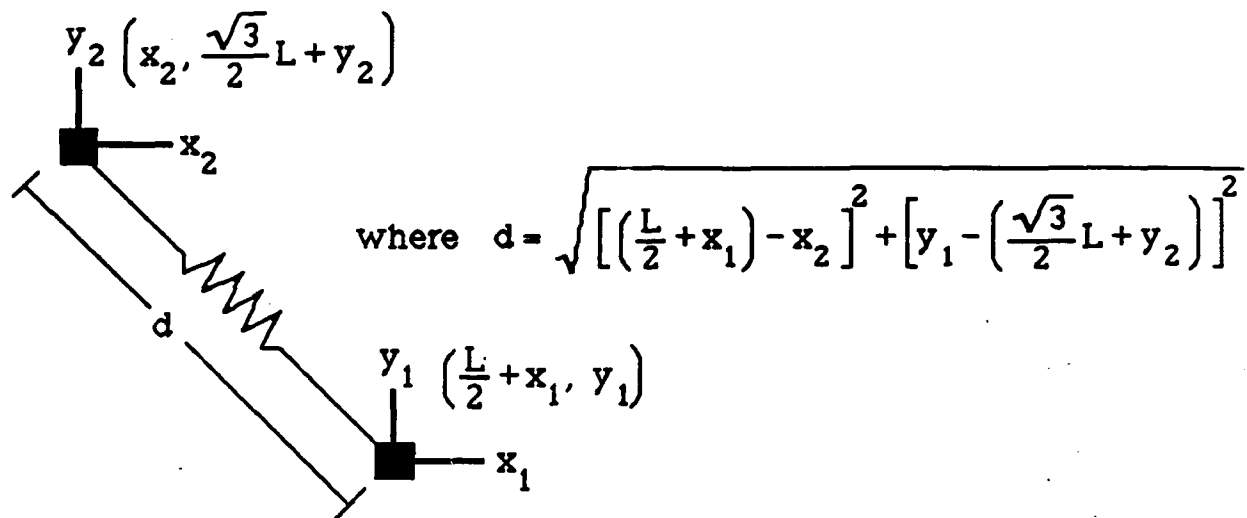


Figure 3.2

The potential energy of this spring is one half the spring constant  $k$  times the square of the distance that the spring is stretched. If we let  $d$  represent the length of the spring after it has been stretched, then the displacement of the spring from its equilibrium position (the distance that the spring is stretched) is  $|d - L|$ . Expressing the potential energy for the spring in terms of  $|d - L|$ , we have

$$V_{12} = \frac{1}{2}k|d - L|^2$$

where the subscript 12 of  $V$  indicates that we are finding the potential energy of the spring that is stretched between the block

with coordinates  $x_1$  and  $y_1$  to the block with coordinates  $x_2$  and  $y_2$ .

Now, we want to rewrite  $V_{12}$  using the variables  $x_1$ ,  $y_1$ ,  $x_2$ , and  $y_2$ .

To do this, we must first simplify the expression for the distance  $d$ .

$$\begin{aligned}
 d &= \left\{ \left[ \left( \frac{L}{2} + x_1 \right) - x_2 \right]^2 + \left[ y_1 - \left( \frac{\sqrt{3}}{2} L + y_2 \right) \right]^2 \right\}^{\frac{1}{2}} \\
 &= \left\{ \left[ \frac{L}{2} + (x_1 - x_2) \right]^2 + \left[ -\frac{\sqrt{3}}{2} L + (y_1 - y_2) \right]^2 \right\}^{\frac{1}{2}} \\
 &= \left\{ \frac{1}{4} L^2 + L(x_1 - x_2) + (x_1 - x_2)^2 + \frac{3}{4} L^2 - \sqrt{3} L(y_1 - y_2) + (y_1 - y_2)^2 \right\}^{\frac{1}{2}} \\
 &= L \left\{ 1 + \frac{1}{L} [(x_1 - x_2) - \sqrt{3}(y_1 - y_2)] + \frac{1}{L^2} [(x_1 - x_2)^2 + (y_1 - y_2)^2] \right\}^{\frac{1}{2}}
 \end{aligned}$$

We want to rewrite the quantity

$$\left\{ 1 + \frac{1}{L} [(x_1 - x_2) - \sqrt{3}(y_1 - y_2)] + \frac{1}{L^2} [(x_1 - x_2)^2 + (y_1 - y_2)^2] \right\}^{\frac{1}{2}}$$

using its Taylor series expansion. The terms of higher powers have been grouped together for convenience.

$$d = L \left\{ 1 + \frac{\frac{1}{2} \left[ (x_1 - x_2) - \sqrt{3}(y_1 - y_2) \right]}{1!} \frac{1}{L} + \text{terms of higher powers} \right\}$$


---

### Exercise 3.1

Verify that the expression above is indeed the Taylor series expansion for the quantity  $f\left(\frac{1}{L}\right)$  (shown below). Hint: write  $f\left(\frac{1}{L}\right)$  in its Taylor series expanded about zero. Recall that  $L$  is the length of the spring in equilibrium, thus  $L \neq 0$ . (Hint: To make it easier to take the derivative of  $f\left(\frac{1}{L}\right)$ , let  $r = \frac{1}{L}$  and find the derivative of  $f(r)$ .)

$$f\left(\frac{1}{L}\right) = \left\{ 1 + \frac{1}{L} \left[ (x_1 - x_2) - \sqrt{3}(y_1 - y_2) \right] + \frac{1}{L^2} \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right] \right\}^{\frac{1}{2}}$$


---

The expression preceding Exercise 3.1 can be simplified by multiplying through by  $L$  and then moving  $L$  to the left side. The resulting quantity is what we want.

$$d - L = \frac{1}{2} \left[ (x_1 - x_2) - \sqrt{3}(y_1 - y_2) \right] + \text{terms of higher powers}$$

This quantity can now be substituted into the formula for the potential energy  $V_{12}$ .

$$\begin{aligned} V_{12} &= \frac{1}{2} k |d - L|^2 \\ &= \frac{k}{2} \left\{ \frac{1}{4} \left[ (x_1 - x_2) - \sqrt{3}(y_1 - y_2) + \text{terms of higher powers} \right]^2 \right\} \\ &= \frac{k}{2} \left\{ \frac{1}{4} \left[ (x_1 - x_2) - \sqrt{3}(y_1 - y_2) \right]^2 + \text{terms of higher powers} \right\} \end{aligned}$$

Recall from an earlier discussion that the Taylor series expansion for potential energy can not have any nonzero linear terms because we are in a system which has an equilibrium configuration. Also, we are only considering small vibrations so we ignore the terms of higher powers. The formula for potential energy  $V_{12}$  is

$$\begin{aligned} V_{12} &= \frac{k}{2} \left\{ \frac{1}{4} \left[ (x_1 - x_2) - \sqrt{3}(y_1 - y_2) \right]^2 \right\} \\ &= \frac{k}{2} \left\{ \frac{x_1^2}{4} + \frac{x_2^2}{4} + \frac{3y_1^2}{4} + \frac{3y_2^2}{4} - \frac{1}{2} x_1 x_2 - \frac{\sqrt{3}}{2} x_1 y_1 + \frac{\sqrt{3}}{2} x_2 y_1 \right. \\ &\quad \left. + \frac{\sqrt{3}}{2} x_1 y_2 - \frac{\sqrt{3}}{2} x_2 y_2 - \frac{3}{2} y_1 y_2 \right\} \end{aligned}$$

---

**Exercise 3.2**

Using Figure 3.3, find  $V_{23}$ . Hint: the procedure is similar to the one used to find  $V_{12}$ .

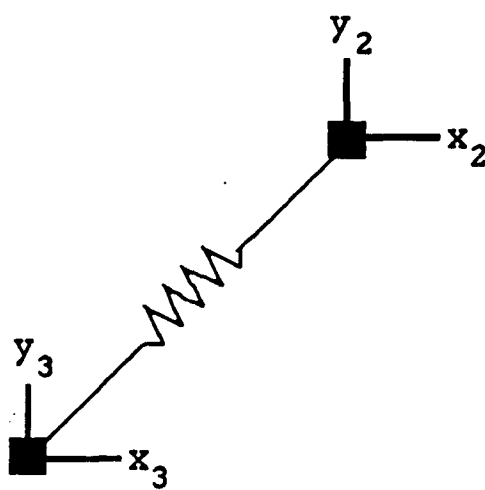


Figure 3.3

**Exercise 3.3**

Using Figure 3.4, find  $V_{13}$ .

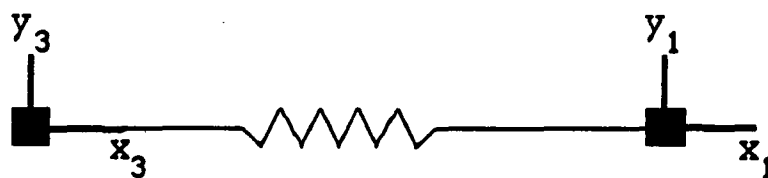


Figure 3.4

---

As stated before, the potential energy of the system is the sum of the potential energy of each spring. Thus we have

$$V = V_{12} + V_{23} + V_{13}$$

$$= \frac{k}{2} \left\{ \frac{5}{4}x_1^2 + \frac{1}{2}x_2^2 + \frac{5}{4}x_3^2 + \frac{3}{4}y_1^2 + \frac{3}{2}y_2^2 + \frac{3}{4}y_3^2 - \frac{x_1x_2}{2} - \frac{x_2x_3}{2} \right. \\ \left. - 2x_1x_3 - \frac{3}{2}y_1y_2 - \frac{3}{2}y_2y_3 - \frac{\sqrt{3}}{2}x_1y_1 + \frac{\sqrt{3}}{2}x_2y_1 + \frac{\sqrt{3}}{2}x_1y_2 \right. \\ \left. - \frac{\sqrt{3}}{2}x_3y_2 - \frac{\sqrt{3}}{2}x_2y_3 + \frac{\sqrt{3}}{2}x_3y_3 \right\}$$

We recall that the kinetic energy is given by

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2)$$

Since this system requires six coordinates to fully describe it, we know we must have six equations of motion. These are

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = -\frac{\partial V}{\partial x_1}, \quad \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = -\frac{\partial V}{\partial x_2}, \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_3} \right] = -\frac{\partial V}{\partial x_3}, \\ \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_4} \right] = -\frac{\partial V}{\partial x_4}, \quad \left[ \frac{\partial T}{\partial \dot{x}_5} \right] = -\frac{\partial V}{\partial x_5}, \quad \text{and} \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_6} \right] = -\frac{\partial V}{\partial x_6}$$

First, we find the left side of each equation of motion, then find the right side and equate the two. Thus, we have the following six equations.

$$\ddot{x}_1 = \frac{k}{m} \left\{ -\frac{5}{4}x_1 + \frac{1}{4}x_2 + x_3 + \frac{\sqrt{3}}{4}y_1 - \frac{\sqrt{3}}{4}y_2 \right\}$$

$$\ddot{x}_2 = \frac{k}{m} \left\{ \frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{1}{4}x_3 - \frac{\sqrt{3}}{4}y_1 + \frac{\sqrt{3}}{4}y_3 \right\}$$

$$\ddot{x}_3 = \frac{k}{m} \left\{ x_1 + \frac{1}{4}x_2 - \frac{5}{4}x_3 + \frac{\sqrt{3}}{4}y_2 - \frac{\sqrt{3}}{4}y_3 \right\}$$

$$\ddot{y}_1 = \frac{k}{m} \left\{ \frac{\sqrt{3}}{4}x_1 - \frac{\sqrt{3}}{4}x_2 - \frac{3}{4}y_1 + \frac{3}{4}y_2 \right\}$$

$$\ddot{y}_2 = \frac{k}{m} \left\{ -\frac{\sqrt{3}}{4}x_1 + \frac{\sqrt{3}}{4}x_3 + \frac{3}{4}y_1 - \frac{3}{2}y_2 + \frac{3}{4}y_3 \right\}$$

$$\ddot{y}_3 = \frac{k}{m} \left\{ \frac{\sqrt{3}}{4}x_2 - \frac{\sqrt{3}}{4}x_3 + \frac{3}{4}y_2 - \frac{3}{4}y_3 \right\}$$

This system of six equations can be written as a matrix equation. In order to eliminate fractions from the matrix, we factor  $\frac{k}{4m}$  out of each equation, which results in  $\frac{k}{4m}$  being factored out of the coefficient matrix A.

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} = \frac{k}{4m} \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{k}{4m} A \vec{X}$$

The generalized theory developed in Exercise 2.3 describes the situation when  $n=6$ . Thus, we begin by finding the eigenvalues and eigenvectors of the symmetric matrix  $A$ . One way we could proceed would be to use the sixth degree characteristic polynomial to find the eigenvalues directly. However, this would require finding the determinant of a  $6 \times 6$  matrix. Using the cofactor expansion method would require  $6!$  or 720 calculations to find the value of the determinant. We could also use a computer program. For example, the user's guide to the computer program LINPACK (Dongarra, Bunch and Stewart, 1979) describes how the program can be used to approximate the eigenvalues and eigenvectors of the characteristic polynomial. This would be quicker, but would not give us any insight into the possible types of vibrations of the system. Instead, let us consider the symmetric matrix  $A$  and see if we can use our knowledge of matrices to reduce the amount of work required to find the eigenvalues. In general, the coefficient matrix which represents an application is much larger than a  $6 \times 6$  matrix, but is still a symmetric matrix. The approach used by applied mathematicians working on large systems would be to: 1) manually work through the theory of a smaller, related, and less complicated system, 2) enlarge the system and use a computer to find the eigenvalues and eigenvectors, 3) interpret the physical meaning of the information from the computer by comparing the results with the results found in step 1, and finally, 4) change the model so that it reflects the desired system as closely as possible. For example, in a more complicated system not all of the blocks may



be of the same mass, nor the springs be of the same length or have the same spring constant. Step 1 may be to consider a system where all of the blocks have the same mass, the springs are all of the same length, and each spring has the same spring constant.

Therefore, we will start our work by finding the determinant of the matrix  $A - \lambda I$

$$\det(A - \lambda I) = \begin{vmatrix} -5 - \lambda & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 - \lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 - \lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 - \lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 - \lambda & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 - \lambda \end{vmatrix}$$

Next, we replace the first row by the sum the first three rows and replace the last row by the sum of the last three rows to obtain the following interesting matrix.

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda & 0 & 0 & 0 \\ 1 & -2 - \lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 - \lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 - \lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 - \lambda & 3 \\ 0 & 0 & 0 & -\lambda & -\lambda & -\lambda \end{vmatrix} = |A'|$$

If  $\lambda$  were set equal to zero, the matrix  $A'$ , defined above, would have two rows of zeros indicating that  $\lambda=0$  is an eigenvalue of  $A$  with multiplicity  $\geq 2$ .

We pause for a moment in our pursuit of eigenvalues to find the eigenvectors associated with  $\lambda=0$ . To begin,  $\lambda=0$  is substituted into  $A'$  so that the matrix equation  $A' \vec{X} = \vec{0}$ , which has been written in augmented form, can be solved.

$$\left( \begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} & 0 \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Using Gaussian elimination, we reduce this system to a form that can easily be solved.

$$(3.1) \quad \left( \begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

From this augmented system of equations, with three rows of zeros, we know three of the variables ( $x_3$ ,  $y_2$  and  $y_3$ ) can take on any value, forcing the remaining three variable ( $x_1$ ,  $x_2$  and  $y_1$ ) to take

on specific values given by the following equations, obtained from the augmented matrix in Equation (3.1)..

$$\begin{aligned}x_1 &= x_3 \\(3.2) \quad x_2 &= x_3 - \sqrt{3}y_2 + \sqrt{3}y_3 \\y_1 &= 2y_2 - y_3\end{aligned}$$

Thus, by letting  $x_3$ ,  $y_2$  and  $y_3$  take on specific values, we will have three linearly independent eigenvectors. This means the eigenvalue  $\lambda=0$  must have multiplicity three.

Before we actually determine the values of the eigenvectors, let us pause for a moment to see how we can rewrite the potential energy function in a slightly different format which will help us to determine its value under certain conditions. Recall, the potential energy of the system is the sum of the potential energy of each spring. Thus we have

$$V = V_{12} + V_{23} + V_{13}$$

$$= \frac{k}{2} \left\{ \frac{5}{4}x_1^2 + \frac{1}{2}x_2^2 + \frac{5}{4}x_3^2 + \frac{3}{4}y_1^2 + \frac{3}{2}y_2^2 + \frac{3}{4}y_3^2 - \frac{x_1x_2}{2} - \frac{x_2x_3}{2} \right. \\ \left. - 2x_1x_3 - \frac{3}{2}y_1y_2 - \frac{3}{2}y_2y_3 - \frac{\sqrt{3}}{2}x_1y_1 + \frac{\sqrt{3}}{2}x_2y_1 + \frac{\sqrt{3}}{2}x_1y_2 \right. \\ \left. - \frac{\sqrt{3}}{2}x_3y_2 - \frac{\sqrt{3}}{2}x_2y_3 + \frac{\sqrt{3}}{2}x_3y_3 \right\}$$

$$= (x_1 \ x_2 \ x_3 \ y_1 \ y_2 \ y_3) \frac{-k}{8m} \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

We are able to rewrite the potential energy in this format because  $A$  is a symmetric matrix. We define a function which can be rewritten in this fashion as a **quadratic form**. Thus, Equation (3.3) is the potential energy expressed as a quadratic form.

$$(3.3) \quad V(\vec{X}) = \vec{X}^T \frac{-k}{8m} A \vec{X}$$

By the definition of an eigenvector  $\vec{X}$ , which is associated with the eigenvalue  $\lambda$  of the matrix  $A$ , we know that  $A \vec{X} = \lambda \vec{X}$ . If

we let the three eigenvectors, associated with the eigenvalue  $\lambda=0$ , be

represented by  $\vec{X}_{\lambda_1=0}$ ,  $\vec{X}_{\lambda_2=0}$ , and  $\vec{X}_{\lambda_3=0}$ , then

$$V(\vec{X}_{\lambda_j=0}) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} \lambda \vec{X}_{\lambda_j=0} = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} 0 \vec{X}_{\lambda_j=0} = 0$$

for  $j=1, 2$  or  $3$ . This tells us that the potential energy is zero. We now examine the physical interpretation of zero potential energy.

To have zero potential energy in the system, all the springs must remain the same length  $L$  as in equilibrium. Thus, the only type of motion possible occurs when the entire system moves as a unit. This is called a **rigid motion**. Since the spring-mass system lies in the  $xy$ -plane, there are only two types of rigid motion: translations (movement in the  $x$ - or  $y$ -direction only) and rotations (the system pivots around its center of mass). These two motions can also be combined.

If we consider the vector  $\vec{X}$ , as a translation in the  $x$ -direction only, then the variables  $x_1$ ,  $x_2$  and  $x_3$  must all change by the same value and the variables  $y_1$ ,  $y_2$  and  $y_3$  can not change. We can

express this vector as  $\begin{pmatrix} c \\ c \\ c \\ 0 \\ 0 \\ 0 \end{pmatrix}$ . If we let  $c=1$ , an eigenvector associated

with the eigenvalue  $\lambda=0$  is  $\vec{X}_{\lambda_1=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ . The graphical

interpretation of  $\vec{X}_{\lambda_1=0}$  can be seen in

Figure 3.5.

**System Motion:**  
Translation along  
the x-axis

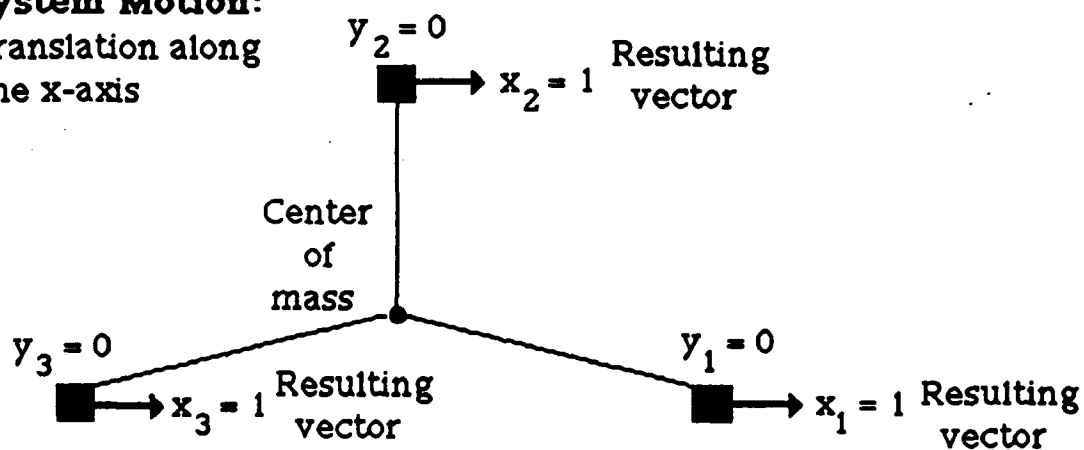


Figure 3.5

If the translation is to the right (in the positive  $x$ -direction), then  $c > 0$  and if it is to the left (in the negative  $x$ -direction), then  $c < 0$ .

---

### Exercise 3.4

Verify that  $\vec{X}_{\lambda_1=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  is a solution to

$$V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$$


---

The other type of translational motion we wish to consider occurs when the system moves in the  $y$ -direction only. The variables  $x_1$ ,  $x_2$  and  $x_3$  do not change while the variables  $y_1$ ,  $y_2$  and  $y_3$  must all change by the same value. We can express this vector

as  $\begin{pmatrix} 0 \\ 0 \\ 0 \\ c \\ c \\ c \end{pmatrix}$ . If we let  $c=1$ , an eigenvector associated with the

eigenvalue  $\lambda=0$  is  $\vec{X}_{\lambda_2=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ . The graphical interpretation of

$\vec{X}_{\lambda_2=0}$  can be seen in

Figure 3.6.

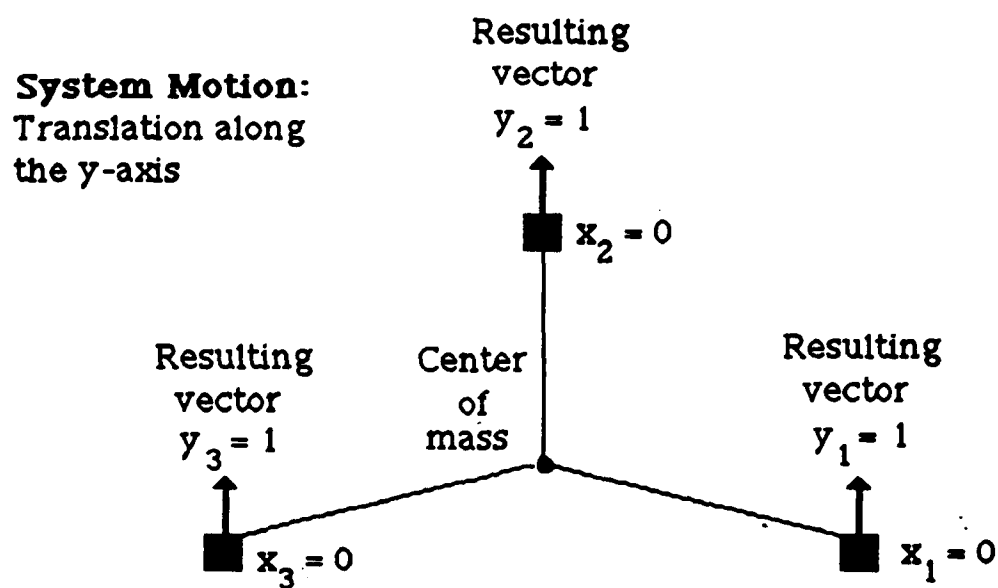


Figure 3.6

If the translation is upward (in the positive y-direction), then  $c > 0$  and if the translation is downward (in the negative y-direction), then  $c < 0$ .



---

**Exercise 3.5**

Verify that  $\vec{X}_{\lambda_2=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$  is a solution to

$$V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0.$$


---

Taking a careful look at the vectors  $\vec{X}_{\lambda_1=0}$  and  $\vec{X}_{\lambda_2=0}$ , we see they are orthogonal. That is, their dot product is zero. Since the eigenspace associated with the eigenvalue  $\lambda=0$  has dimension at least three, we know there is a third linearly independent eigenvector associated with the eigenvalue  $\lambda=0$ . There are two ways we could proceed at this point. The first is to use the three equations in (3.2) and choose values for  $x_3$ ,  $y_2$  and  $y_3$ . For example, let  $x_3=0$ ,  $y_2=1$  and  $y_3=0$ , then use the Gram-Schmidt

process to find a vector which is orthogonal to both  $\vec{X}_{\lambda_1=0}$  and

$\vec{X}_{\lambda_2=0}$ . The other way is to replace two of the rows of zeros in the

coefficient matrix in Equation (3.1) by the eigenvectors  $\vec{X}_{\lambda_1=0}^T$  and

$\vec{X}_{\lambda_2=0}^T$ . The solution to this new augmented matrix must satisfy all

the equations which form the augmented matrix. Hence, the solution to the augmented system will satisfy both  $x_1 + x_2 + x_3 = 0$  (from

$\vec{X}_{\lambda_1=0}$ ) and  $y_1 + y_2 + y_3 = 0$  (from  $\vec{X}_{\lambda_2=0}$ ). A vector whose entries

satisfy both of these equations is orthogonal to the eigenvectors

$\vec{X}_{\lambda_1=0}$  and  $\vec{X}_{\lambda_2=0}$ . Also, from these two equations, we see in the

solution to the equations associated with the augmented matrix, the  $x_i$  values must sum to zero. Therefore, there is no translational

motion in the x-direction. Similarly, there is no translational motion in the y-direction. Thus, the center of mass does not move. Since this motion is a rigid motion ( $\lambda=0$ ) and the center of mass of the system does not move, the rigid motion must be a rotation.

Reducing the following augmented matrix which is Equation(3.1)

with two of its rows of zeros replaced by  $\vec{X}_{\lambda_1=0}^T$  and  $\vec{X}_{\lambda_2=0}^T$

$$\left( \begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

we obtain

$$\left( \begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{2\sqrt{3}}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

This augmented matrix above, can be interpreted as the following equations.

$$x_1 = -\frac{\sqrt{3}}{3}y_3$$

$$x_2 = \frac{2\sqrt{3}}{3}y_3$$

$$x_3 = -\frac{\sqrt{3}}{3}y_3$$

$$y_1 = -y_3$$

$$y_2 = 0$$

If we let  $y_3=3$ , then the resulting eigenvector is  $\vec{X}_{\lambda_3=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$ .

The graphical interpretation of  $\vec{X}_{\lambda_3=0}$  can be seen in Figure 3.7.

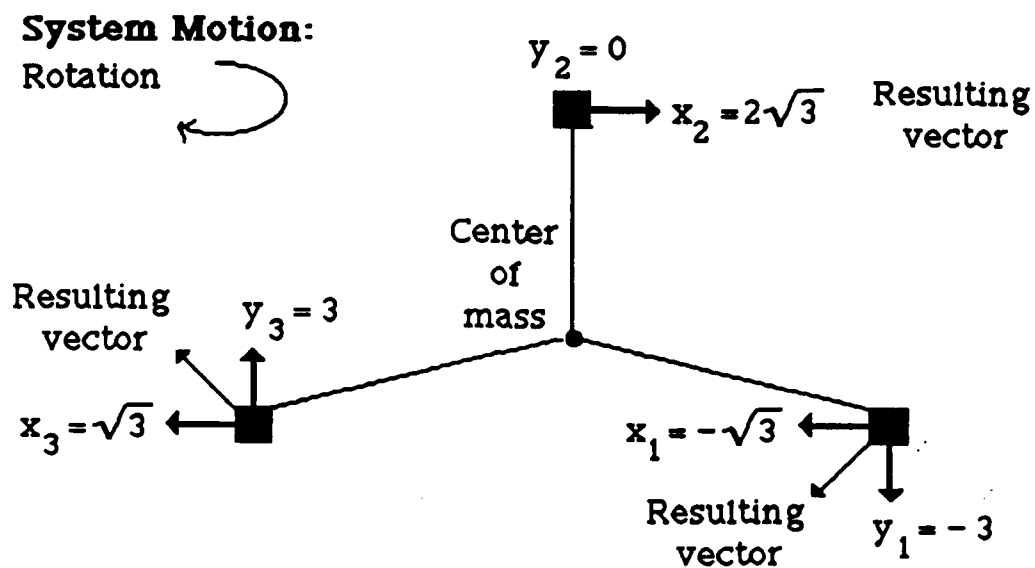


Figure 3.7

## Exercise 3.6

Verify that  $\vec{X}_{\lambda_3=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$  is a solution to

$$V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$$

Also, verify that  $\vec{X}_{\lambda_3=0}$  is orthogonal to both  $\vec{X}_{\lambda_1=0}$  and  $\vec{X}_{\lambda_2=0}$ .

So far we have found only three eigenvalues and their associated eigenvectors. The remaining three eigenvalues can be found using the determinant of the matrix  $A - \lambda I$  which can be reduced to  $|A'|$ . For convenience,  $|A'|$  has been repeated below.

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5-\lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3-\lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6-\lambda & 3 \\ 0 & 0 & 0 & -\lambda & -\lambda & -\lambda \end{vmatrix} = |A'|$$

Using Gaussian elimination, we will reduce the matrix to a form which will make the determinant easier to find. We will use only row (or column) operations that do not change the value of the determinant. After several row operations, we obtain

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3-\lambda & 0 & -2\sqrt{3} & -\sqrt{3} & 0 \\ 0 & -3 & -9-\lambda & \sqrt{3} & 2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} & -\sqrt{3} & -3-\lambda & 3 & 0 \\ 0 & -3\sqrt{3} & 0 & -6-2\lambda & -3-\lambda & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

At this point, we could find the determinant using the cofactor expansion method. However, if we do one column operation we will greatly reduce the number of calculations needed. We add -2 times the fifth column to the fourth column producing a new fourth column.

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3-\lambda & 0 & 0 & -\sqrt{3} & 0 \\ 0 & -3 & -9-\lambda & -3\sqrt{3} & 2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} & -\sqrt{3} & -9-\lambda & 3 & 0 \\ 0 & -3\sqrt{3} & 0 & 0 & -3-\lambda & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

We are now ready to use the cofactor expansion method to find the determinant of the matrix  $A'$ . Expanding by the first column we have

$$|A'| = 1(-1)^{1+1} \begin{vmatrix} -3-\lambda & 0 & 0 & -\sqrt{3} & 0 \\ -3 & -9-\lambda & -3\sqrt{3} & 2\sqrt{3} & 0 \\ -2\sqrt{3} & -\sqrt{3} & -9-\lambda & 3 & 0 \\ -3\sqrt{3} & 0 & 0 & -3-\lambda & 0 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

Now, expand the resulting cofactor by the fifth column and obtain

$$|A'| = 1 \cdot 1(-1)^{5+5} \begin{vmatrix} -3-\lambda & 0 & 0 & -\sqrt{3} \\ -3 & -9-\lambda & -3\sqrt{3} & 2\sqrt{3} \\ -2\sqrt{3} & -\sqrt{3} & -9-\lambda & 3 \\ -3\sqrt{3} & 0 & 0 & -3-\lambda \end{vmatrix}$$

Next, expand the resulting cofactor by the first row and obtain

$$\begin{aligned}
 |A'| &= 1 \cdot 1 \left[ (-3-\lambda)(-1)^{1+1} \begin{vmatrix} -9-\lambda & -3\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & -9-\lambda & 3 \\ 0 & 0 & -3-\lambda \end{vmatrix} \right. \\
 &\quad \left. + -\sqrt{3}(-1)^{1+2} \begin{vmatrix} -3 & -9-\lambda & -3\sqrt{3} \\ -2\sqrt{3} & -\sqrt{3} & -9-\lambda \\ -3\sqrt{3} & 0 & 0 \end{vmatrix} \right] \\
 &= [(-9-\lambda)^2 - 9][(-3-\lambda)^2 - 9].
 \end{aligned}$$

To find the eigenvalues of the original matrix, we set each factor equal to zero and solve for  $\lambda$ .

$$(-9-\lambda)^2 - 9 = 0$$

$$(-9-\lambda)^2 = 9$$

$$-9-\lambda = \pm 3$$

$$\lambda = -12, -6$$

$$(-3-\lambda)^2 - 9 = 0$$

$$(-3-\lambda)^2 = 9$$

$$-3-\lambda = \pm 3$$

$$\lambda = -6, 0$$

Since we have already determined that the eigenvalue  $\lambda=0$  has multiplicity of at least three, the fact  $\lambda=0$  occurs above should be no surprise. The remaining eigenvalues for the matrix  $A$  are  $\lambda=-12$  and  $\lambda=-6$ , the latter with multiplicity 2.



To help us find the associated eigenvectors for the remaining three eigenvalues, we recall the equation  $A \vec{X}_\lambda = \lambda \vec{X}_\lambda$  from the definition of an eigenvector  $\vec{X}_\lambda$ , and Equation 3.3

$V(\vec{X}_\lambda) = \vec{X}_\lambda^T \frac{-k}{8m} A \vec{X}_\lambda$ , which describes the potential energy as a quadratic form using eigenvectors. As we saw earlier, these two equations can be combined as  $V(\vec{X}_\lambda) = \vec{X}_\lambda^T \frac{-k}{8m} \lambda \vec{X}_\lambda$ . We

observe that the only way this equation can equal zero is if  $\lambda=0$  or  $\vec{X}_\lambda$  is the zero vector. However, since we are only looking at  $\lambda=-12$

or  $\lambda=-6$ , which are nonzero values, we must have that  $\vec{X}_\lambda$  be the

zero vector in order for  $V(\vec{X}_\lambda)$  to equal zero. Clearly this cannot

happen because  $\vec{X}_\lambda$  is an eigenvector which by definition is never equal to the zero vector. This indicates that the potential energy of the system is not zero. Hence, the potential energy of each spring is not zero, so the length of at least one of the springs must change.

Thus, we do not have a rigid motion. Also, we recall that the determinant of the matrix  $A-\lambda I$  can be reduced by summing the first three rows and the last three rows to give  $|A'|$ .

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5-\lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3-\lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6-\lambda & 3 \\ 0 & 0 & 0 & -\lambda & -\lambda & -\lambda \end{vmatrix} = |A'|$$

If we substitute in  $\lambda = -12$  or  $\lambda = -6$ , the first row will contain constant values for  $x_1$ ,  $x_2$ , and  $x_3$  and the last row will contain constant

values for  $y_1$ ,  $y_2$ , and  $y_3$ . From an earlier discussion (following

Exercise 3.5) this indicates there is no translational motion in either the  $x$ - or  $y$ -directions, so we know the center of mass does not move. Thus the motion associated with the last three eigenvectors can be thought of as vibrations of the blocks (but not a translation or rotation) with the center of mass remaining fixed.

First, we find the two linearly independent eigenvectors associated with the eigenvalue  $\lambda = -6$ . Since  $\lambda = -6$  has multiplicity two, the solution space of the augmented matrix,  $(A - \lambda I) \vec{X} = \vec{0}$  or  $(A + 6I) \vec{X} = \vec{0}$  will have dimension four. That is, when the augmented system is reduced, we will have two rows of zeros. Thus, four of the variables can be written in terms of two of the other variables. These two variables can be assigned values which will produce two linearly independent eigenvectors. If we let  $x_2$  and  $y_2$  be these two variables, then  $x_1$ ,  $y_1$ ,  $x_3$  and  $y_3$  can be written

in terms of  $x_2$  and  $y_2$ . One way to assign values to  $x_2$  and  $y_2$  and be assured of getting a linearly independent eigenvector, is to first let  $x_2=0$  and  $y_2=1$ , and then let  $x_2=1$  and  $y_2=0$ . Let us consider the geometric interpretation of these cases.

CASE 1.  $x_2=0$  and  $y_2=1$

Since the center of mass for this configuration remains fixed, the  $y_2$  component must be balanced by the sum of the  $y_1$  and the  $y_3$

components. Because  $x_2=0$ , we know the components  $x_1$  and  $x_3$

must be equal in magnitude and of opposite sign. These components can be seen in Figure 3.8.

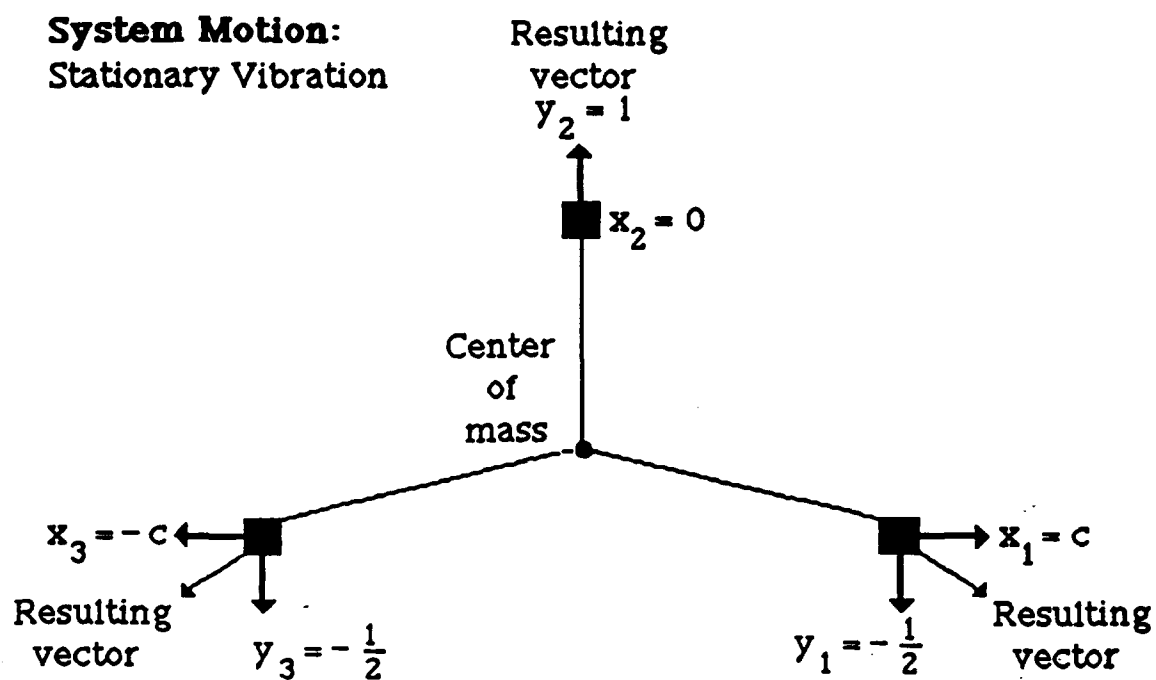


Figure 3.8

Letting  $c=1$ , one eigenvector associated with  $\lambda=-6$  is

$$\vec{X}_{\lambda_1=-6} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

CASE 2.  $x_2=1$  and  $y_2=0$

Since the center of mass does not move, the  $x_2$  component is balanced by the sum of the  $x_1$  and  $x_3$  components. Because  $y_2=0$ , we know the components  $y_1$  and  $y_3$  must be equal in magnitude and of opposite sign. These components can be seen in Figure 3.9.

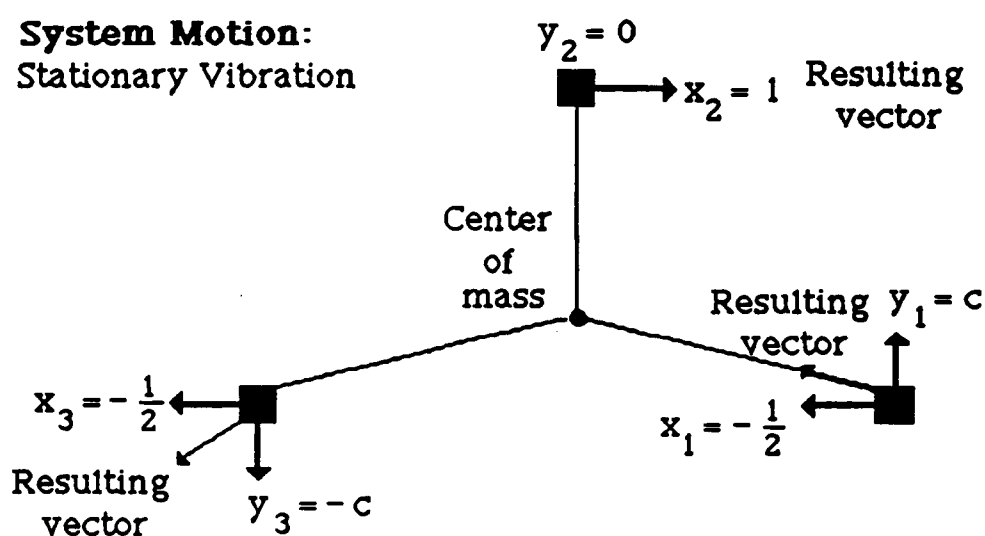


Figure 3.9

Letting  $c=1$ , a second eigenvector associated with  $\lambda=-6$  is

$$\vec{x}_{\lambda_2=-6} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

---

**Exercise 3.7**

Show that  $\vec{X}_{\lambda_1=-6}$  and  $\vec{X}_{\lambda_2=-6}$  are orthogonal (their dot product is zero).

---

Therefore,  $\{ \vec{X}_{\lambda_1=-6}, \vec{X}_{\lambda_2=-6} \}$  is a set of orthogonal eigenvectors associated with the eigenvalue  $\lambda=-6$ .

It remains for us to find the single eigenvector associated with the eigenvalue  $\lambda=-12$ . To do this we will substitute  $-12$  for  $\lambda$  in  $|A|$  and solve the matrix equation  $A' \vec{X} = \vec{0}$ . When we do this, we get the following augmented matrix.

$$\left( \begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

This augmented matrix above, can be interpreted as the following equations.

$$x_1 = -\sqrt{3}y_3$$

$$x_2 = 0$$

$$x_3 = \sqrt{3}y_3$$

$$y_1 = y_3$$

$$y_2 = -2y_3$$

If we let  $y_3=1$ , then the resulting eigenvector is  $\vec{X}_{\lambda=-12} = \begin{pmatrix} -\sqrt{3} \\ 0 \\ \sqrt{3} \\ 1 \\ -2 \\ 1 \end{pmatrix}$ .

The graphical interpretation of  $\vec{X}_{\lambda=-12}$  can be seen in Figure 3.10.

**System Motion:**  
Stationary Vibration

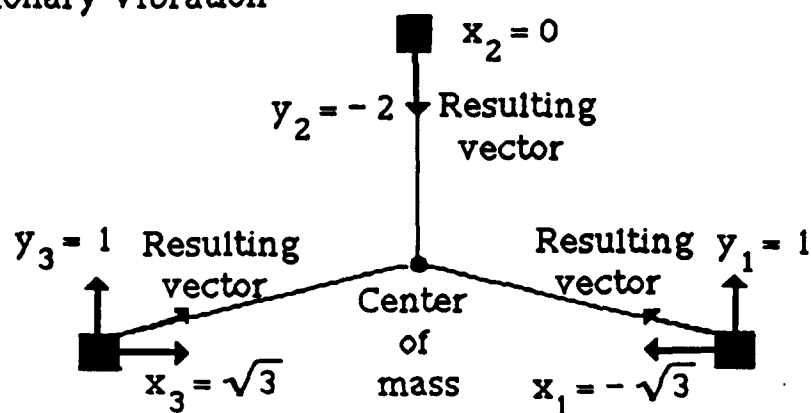


Figure 3.10

$$\text{Thus, } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1/2 \\ 1 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ 0 \\ \sqrt{3} \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\} \text{ is an}$$

orthogonal set of eigenvectors of  $A$  associated with the eigenvalues  $0, 0, 0, -6, -6$  and  $-12$ , respectively. We normalize these orthogonal vectors to get the following orthonormal set of eigenvectors.



$$\left\{ \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{6} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{14}}{7} \\ 0 \\ -\frac{\sqrt{14}}{7} \\ -\frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ -\frac{\sqrt{14}}{14} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ -\frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ 0 \\ -\frac{\sqrt{14}}{7} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{6} \end{pmatrix} \right\}$$

If we let the eigenvectors above form the columns of a matrix  $P$ , then  $P$  is invertible and  $P^{-1}AP = D$ , so  $P^{-1} \frac{k}{4m} AP = \frac{k}{4m} D$  where  $D$  is a

diagonal matrix with the eigenvalues 0, 0, 0, -6, -6, and -12 as the entries on the diagonal. Since our goal is to solve the differential

equation  $\ddot{\vec{X}} = \frac{k}{4m} A \vec{X}$ , we will let  $\vec{U} = P^{-1} \vec{X}$ , then apply Exercise

2.4 with  $n=6$ , where we have factored  $\frac{k}{4m}$  out of the matrix  $A$

instead of  $\frac{k}{m}$ . Thus  $P \ddot{\vec{U}} = \frac{k}{4m} P D \vec{U}$  becomes

$$P^{(1)} \ddot{u}_1 + P^{(2)} \ddot{u}_2 + \dots + P^{(6)} \ddot{u}_6 = \frac{k}{4m} [P^{(1)} \lambda_1 u_1 + P^{(2)} \lambda_2 u_2 + \dots + P^{(6)} \lambda_6 u_6]$$

where  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ ,  $\lambda_4 = \lambda_5 = -6$ , and  $\lambda_6 = -12$ . This equation can be

simplified by multiplying through by  $\frac{k}{4m}$ , gathering terms and

moving everything to the left side.

$$\begin{aligned} & \left[ P^{(1)} \ddot{u}_1 - \frac{k}{4m} P^{(1)} \lambda_1 u_1 \right] + \left[ P^{(2)} \ddot{u}_2 - \frac{k}{4m} P^{(2)} \lambda_2 u_2 \right] + \dots \\ & \quad + \left[ P^{(6)} \ddot{u}_6 - \frac{k}{4m} P^{(6)} \lambda_6 u_6 \right] = \vec{0} \end{aligned}$$

Factoring out  $P^{(1)}, P^{(2)}, \dots, P^{(6)}$ , we have

$$\left[ \ddot{u}_1 - \frac{k}{4m} \lambda_1 u_1 \right] P^{(1)} + \left[ \ddot{u}_2 - \frac{k}{4m} \lambda_2 u_2 \right] P^{(2)} + \dots + \left[ \ddot{u}_6 - \frac{k}{4m} \lambda_6 u_6 \right] P^{(6)} = \vec{0}$$

Since the columns of  $P$  are orthonormal eigenvectors of  $A$ , we know they are linearly independent. Thus, we have a finite linear combination of linearly independent vectors which equals zero, so the coefficients of  $P^{(1)}, P^{(2)}, \dots, P^{(6)}$  must be zero. If we set each of

the coefficients in the above equation equal to zero, we have

$$\begin{array}{ll} \ddot{u}_1 - \frac{k}{4m} \lambda_1 u_1 = 0 & \ddot{u}_1 = \frac{k}{4m} \lambda_1 u_1 \\ \ddot{u}_2 - \frac{k}{4m} \lambda_2 u_2 = 0 & \text{or } \ddot{u}_2 = \frac{k}{4m} \lambda_2 u_2 \\ \vdots & \vdots \\ \ddot{u}_6 - \frac{k}{4m} \lambda_6 u_6 = 0 & \ddot{u}_6 = \frac{k}{4m} \lambda_6 u_6 \end{array}$$

Since  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ ,  $\lambda_4 = \lambda_5 = -6$  and  $\lambda_6 = -12$ , these differential equations become

$$\ddot{u}_1 = 0$$

$$\ddot{u}_4 = -\frac{3k}{2m} u_4$$

$$\ddot{u}_2 = 0$$

$$\ddot{u}_5 = -\frac{3k}{2m} u_5$$

$$\ddot{u}_3 = 0$$

$$\ddot{u}_6 = -\frac{3k}{m} u_6$$

These are all second order linear differential equations which can be solved using basic techniques. (See Appendix A.) The solutions are

$$u_1 = c_{11}t + c_{12}$$

$$u_2 = c_{21}t + c_{22}$$

$$u_3 = c_{31}t + c_{32}$$

$$u_4 = c_{41} \cos\left(\sqrt{\frac{3k}{2m}} t\right) + c_{42} \sin\left(\sqrt{\frac{3k}{2m}} t\right)$$

$$u_5 = c_{51} \cos\left(\sqrt{\frac{3k}{2m}} t\right) + c_{52} \sin\left(\sqrt{\frac{3k}{2m}} t\right)$$

$$u_6 = c_{61} \cos\left(\sqrt{\frac{3k}{m}} t\right) + c_{62} \sin\left(\sqrt{\frac{3k}{m}} t\right)$$

These solutions can be written as a matrix equation which can be substituted into  $\vec{X} = P \vec{U}$ .

$$\vec{X} = P \vec{U}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{6} & \frac{\sqrt{14}}{7} & \frac{\sqrt{14}}{14} & -\frac{1}{2} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{14}}{7} & 0 \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{14}}{7} & -\frac{\sqrt{14}}{14} & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{1}{2} & -\frac{\sqrt{14}}{14} & \frac{\sqrt{14}}{7} & \frac{\sqrt{3}}{6} \\ 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{14}}{7} & 0 & -\frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} & \frac{1}{2} & -\frac{\sqrt{14}}{14} & -\frac{\sqrt{14}}{7} & \frac{\sqrt{3}}{6} \end{pmatrix} \begin{pmatrix} c_{11}t + c_{12} \\ c_{21}t + c_{22} \\ c_{31}t + c_{32} \\ c_{41} \cos\left(\sqrt{\frac{3k}{2m}} t\right) + c_{42} \sin\left(\sqrt{\frac{3k}{2m}} t\right) \\ c_{51} \cos\left(\sqrt{\frac{3k}{2m}} t\right) + c_{52} \sin\left(\sqrt{\frac{3k}{2m}} t\right) \\ c_{61} \cos\left(\sqrt{\frac{3k}{m}} t\right) + c_{62} \sin\left(\sqrt{\frac{3k}{m}} t\right) \end{pmatrix}$$

The solution  $\vec{X}$  to  $\ddot{\vec{X}} = \frac{k}{4m} A \vec{X}$  is found by multiplying the matrix  $P$  by the vector  $\vec{U}$ . The components of the solution vector  $\vec{X}$  are

$$\begin{aligned} x_1 = & c_{11} \frac{\sqrt{3}}{3} t + c_{12} \frac{\sqrt{3}}{3} - c_{31} \frac{\sqrt{3}}{6} t - c_{32} \frac{\sqrt{3}}{6} + c_{41} \frac{\sqrt{14}}{7} \cos\left(\sqrt{\frac{3k}{2m}} t\right) \\ & + c_{42} \frac{\sqrt{14}}{7} \sin\left(\sqrt{\frac{3k}{2m}} t\right) - c_{51} \frac{\sqrt{14}}{14} \cos\left(\sqrt{\frac{3k}{2m}} t\right) \\ & - c_{52} \frac{\sqrt{14}}{14} \sin\left(\sqrt{\frac{3k}{2m}} t\right) - c_{61} \frac{1}{2} \cos\left(\sqrt{\frac{3k}{m}} t\right) - c_{62} \frac{1}{2} \sin\left(\sqrt{\frac{3k}{m}} t\right) \\ x_2 = & c_{11} \frac{\sqrt{3}}{3} t + c_{12} \frac{\sqrt{3}}{3} + c_{31} \frac{\sqrt{3}}{3} t + c_{32} \frac{\sqrt{3}}{3} + c_{51} \frac{\sqrt{14}}{7} \cos\left(\sqrt{\frac{3k}{2m}} t\right) \\ & - c_{52} \frac{\sqrt{14}}{7} \sin\left(\sqrt{\frac{3k}{2m}} t\right) \end{aligned}$$

$$\begin{aligned}
 x_3 = & c_{11} \frac{\sqrt{3}}{3} t + c_{12} \frac{\sqrt{3}}{3} - c_{31} \frac{\sqrt{3}}{6} t - c_{32} \frac{\sqrt{3}}{6} - c_{41} \frac{\sqrt{14}}{7} \cos \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & - c_{42} \frac{\sqrt{14}}{7} \sin \left( \sqrt{\frac{3k}{2m}} t \right) - c_{51} \frac{\sqrt{14}}{14} \cos \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & - c_{52} \frac{\sqrt{14}}{14} \sin \left( \sqrt{\frac{3k}{2m}} t \right) + c_{61} \frac{1}{2} \cos \left( \sqrt{\frac{3k}{m}} t \right) + c_{62} \frac{1}{2} \sin \left( \sqrt{\frac{3k}{m}} t \right)
 \end{aligned}$$

$$\begin{aligned}
 y_1 = & c_{21} \frac{\sqrt{3}}{3} t + c_{22} \frac{\sqrt{3}}{3} - c_{31} \frac{1}{2} t - c_{32} \frac{1}{2} - c_{41} \frac{\sqrt{14}}{14} \cos \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & - c_{42} \frac{\sqrt{14}}{14} \sin \left( \sqrt{\frac{3k}{2m}} t \right) + c_{51} \frac{\sqrt{14}}{7} \cos \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & + c_{52} \frac{\sqrt{14}}{7} \sin \left( \sqrt{\frac{3k}{2m}} t \right) + c_{61} \frac{\sqrt{3}}{6} \cos \left( \sqrt{\frac{3k}{m}} t \right) + c_{62} \frac{\sqrt{3}}{6} \sin \left( \sqrt{\frac{3k}{m}} t \right)
 \end{aligned}$$

$$\begin{aligned}
 y_2 = & c_{21} \frac{\sqrt{3}}{3} t + c_{22} \frac{\sqrt{3}}{3} + c_{41} \frac{\sqrt{14}}{7} \cos \left( \sqrt{\frac{3k}{2m}} t \right) + c_{42} \frac{\sqrt{14}}{7} \sin \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & - c_{61} \frac{\sqrt{3}}{3} \cos \left( \sqrt{\frac{3k}{m}} t \right) - c_{62} \frac{\sqrt{3}}{3} \sin \left( \sqrt{\frac{3k}{m}} t \right)
 \end{aligned}$$

$$\begin{aligned}
 y_3 = & c_{21} \frac{\sqrt{3}}{3} t + c_{22} \frac{\sqrt{3}}{3} + c_{31} \frac{1}{2} t + c_{32} \frac{1}{2} - c_{41} \frac{\sqrt{14}}{4} \cos \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & - c_{42} \frac{\sqrt{14}}{14} \sin \left( \sqrt{\frac{3k}{2m}} t \right) - c_{51} \frac{\sqrt{14}}{7} \cos \left( \sqrt{\frac{3k}{2m}} t \right) \\
 & - c_{52} \frac{\sqrt{14}}{7} \sin \left( \sqrt{\frac{3k}{2m}} t \right) + c_{61} \frac{\sqrt{3}}{6} \cos \left( \sqrt{\frac{3k}{m}} t \right) + c_{62} \frac{\sqrt{3}}{6} \sin \left( \sqrt{\frac{3k}{m}} t \right)
 \end{aligned}$$

It is possible to determine the values of the  $c_{ij}$  provided we have been given enough details about the system which we have modeled. Since the values of  $m$  (the mass of the spring) and  $L$  (the length of the spring in equilibrium) are given, and if we know the value of each  $c_{ij}$ , then we will be able to find the value for each  $x_i$  and  $y_i$  at a given time.

We now want to apply the theory of the spring-mass system which we have just studied to understand how this system models the vibrations of a water or  $H_2O$  molecule, as shown in Figure 3.11.

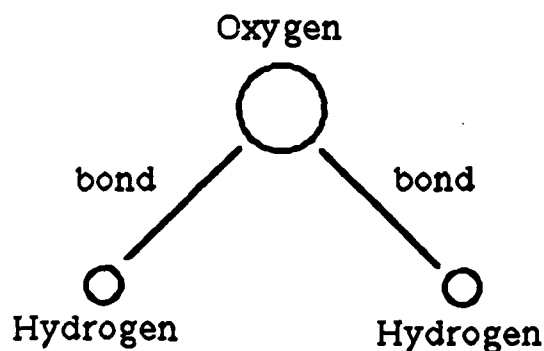


Figure 3.11

Figure 3.1 has three springs, but the water molecule has only two bonds. The third spring represents the repulsion force of the two hydrogen atoms. A water molecule which lies in the  $xy$ -plane would have a translational motion in both the  $x$ - and  $y$ -directions as described in Figures 3.5 and 3.6. Also, the molecule would be able

to rotate, as we saw in Figure 3.7. Moncrief and Jones (1977) explain the three vibrational modes for  $\text{H}_2\text{O}$  using Figure 3.12.

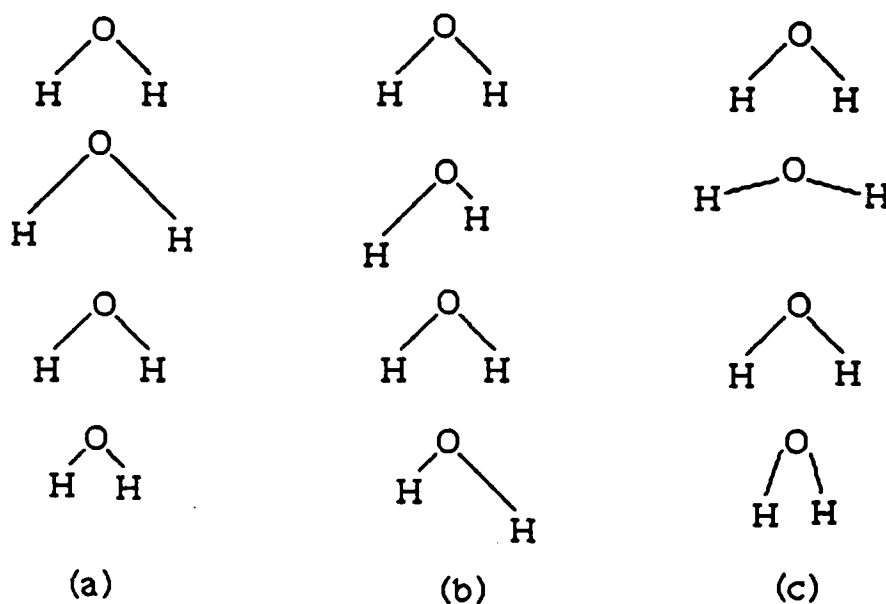


Figure 3.12

The vibration in Figure 3.12(a) is called a symmetric stretch since the bonds between the both hydrogen atoms and the oxygen atom are stretching by the same amounts at the same time. We have already considered this type of motion in Figure 3.8. In Figure 3.12(b) we see an asymmetric stretch which is due to the fact the bonds between the hydrogen atoms and the oxygen atom are being stretched by the same amount but not at the same time. Figure 3.9 describes this same mode of vibration in the closed spring-mass system. The final mode of vibration is called symmetric bending and is seen in Figure 3.12(c). This vibrational mode consists of the

hydrogen-oxygen bonds remaining at the same length, but the two hydrogen atoms vibrate by moving further apart then closer together. We have already seen this in Figure 3.10.



## References

Dongarra, J., Bunch, J., & Stewart, C. (1979). LINPACK user's guide [Computer program manual]. Philadelphia, PA: SIAM.

Moncrief, J. W., & Jones, W. H. (1977). Elements of physical chemistry. Reading, MA: Addison-Wesley.

## Appendix A: Review of Differential Equations

We will limit our discussion to second order linear differential equations with constant coefficients. This appendix is not meant to replace a differential equation course, but only to show how to solve a very select group of differential equations. The second order linear differential equations which we want to solve are of the form

$$(A.1) \quad \ddot{x} + m x = 0,$$

where the coefficient of the  $x$ -term is a constant which we denote by  $m$ . Any second order differential equation which can be put in the form of Equation (A.1) is called a **linear differential equation**. The differential equation  $\ddot{x} + m \sin x = 0$  is no longer linear because  $\sin x$  is a nonlinear function of  $x$ . The method used to solve the differential equation (A.1) above, depends on the value of  $m$ . We will consider three possible cases.

### CASE 1. $m=0$

If  $m=0$ , then our second order differential equation becomes

$$\ddot{x} = 0.$$

By the Fundamental Theorem of Calculus, if  $\ddot{x} = 0$ , then  $\dot{x} = a$ , and  $x = at + b$ .

Conversely, if  $x = at + b$ , then differentiating this equation with respect to time we have

$$\frac{dx}{dt} = a \quad \text{or} \quad \dot{x} = a$$

Differentiating again, we obtain

$$\frac{d^2x}{dt^2} = 0 \quad \text{or} \quad \ddot{x} = 0$$

Therefore, we conclude that  $x = at + b$  is the solution to the differential equation  $\ddot{x} = 0$ .

#### CASE 2. $m < 0$

The differential equation  $\ddot{x} + m x = 0$  can be rearranged as  $\ddot{x} = -m x$  where  $-m$  is a positive number. Recall from calculus, that the exponential function, when differentiated, yields a multiple of itself. Thus, we want an exponential function which when differentiated twice results in a positive multiple of itself. Let us pause for a moment and consider two examples of exponential functions.

$$x = e^{2t} \quad \text{and} \quad x = e^{-2t}$$

Taking the first derivative of these two function with respect to time, we obtain

$$\dot{x} = 2e^{2t} \quad \text{and} \quad \dot{x} = -2e^{-2t}$$

After taking the second derivative, we have the following two functions

$$\ddot{x} = 4e^{2t} \quad \text{and} \quad \ddot{x} = 4e^{-2t}$$

If we substitute the values for  $\ddot{x}$  and  $x$  into the differential equation  $\ddot{x} - 4x = 0$ , we see that  $x = e^{2t}$  and  $x = e^{-2t}$  are both solutions to the same differential equation. Furthermore, any linear combination of these two solutions such as  $x = c_1 e^{2t} + c_2 e^{-2t}$ , is also a solution to

$\ddot{x} + mx = 0$  when  $m = -4$ . From this we conclude that

$x = c_1 e^{\sqrt{-m}t} + c_2 e^{-\sqrt{-m}t}$ , is a solution, for all  $c_1$  and  $c_2$ .

### CASE 3. $m > 0$

The differential equation  $\ddot{x} + mx = 0$  can be rearranged as  $\ddot{x} = -mx$  where  $-m$  is a negative number. Recall from calculus, that the cosine function, when differentiated twice yields a negative multiple of itself. This is also true for the sine function. Let us pause for a moment and consider two examples involving the cosine and sine functions.

$$x = \cos 2x \quad \text{and} \quad x = \sin 2x$$

Taking the first derivative of these two function with respect to time, we obtain

$$\dot{x} = -2\sin 2x \quad \text{and} \quad \dot{x} = 2\cos 2x.$$

After taking the second derivative, we have the following two functions

$$\ddot{x} = -4\cos 2x \quad \text{and} \quad \ddot{x} = -4\sin 2x.$$

If we substitute the values for  $\ddot{x}$  and  $x$  into the differential equation  $\ddot{x} + 4x = 0$ , we see that  $x = \cos 2x$  and  $x = \sin 2x$  are both solutions to the same differential equation. Furthermore, any linear combination of these two solutions such as  $x = c_1 \cos 2x + c_2 \sin 2x$ , is also a solution to  $\ddot{x} + mx = 0$  when  $m=4$ . From this we conclude that  $x = c_1 \cos \sqrt{m} t + c_2 \sin \sqrt{m} t$ , is a solution, for all  $c_1$  and  $c_2$ .

Just as it was shown in Case 1, where  $m=0$ , every solution of  $\ddot{x} = 0$  must be in the form  $at+b$ . It can also be shown that every solution for Cases 2 and 3, where  $m \neq 0$ , must be in the forms we have presented.

## Appendix B: Solutions to Exercises

Exercise 2.1

The system  $\vec{\ddot{U}} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{k}{m} D \vec{U}$  can be rewritten as the following two second order differential equations

$$\begin{aligned} \ddot{u}_1 &= \frac{-3k}{m} u_1 \\ \ddot{u}_2 &= \frac{-k}{m} u_2 \end{aligned} \quad (\text{B.1})$$

Similarly, the system  $\vec{\ddot{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m} A \vec{X}$  can be rewritten as the following two second order differential equations

$$\begin{aligned} \ddot{x}_1 &= \frac{-2k}{m} x_1 + \frac{k}{m} x_2 \\ \ddot{x}_2 &= \frac{k}{m} x_1 - \frac{2k}{m} x_2 \end{aligned} \quad (\text{B.2})$$

Each equation in (B.1) can be solved independently using only basic techniques from differential equations. However, since each equation in (B.2) is in terms of both variables  $x_1$  and  $x_2$ , neither equation can be solved independently. Thus, it is much easier to

solve the system of differential equations given by

$$\vec{\ddot{U}} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{k}{m} D \vec{U}.$$

### Exercise 2.2

We have already found the eigenvalues of  $A$  to be  $\lambda_1 = -3$  and  $\lambda_2 = -1$ ,

thus,  $r_1 = \frac{3k}{m}$  and  $r_2 = \frac{k}{m}$ , which are both greater than zero. From

this we see that the two second order differential equations in  $u_1$  and  $u_2$  are

$$\ddot{u}_1 + \frac{3k}{m} u_1 = 0 \quad \text{and} \quad \ddot{u}_2 + \frac{k}{m} u_2 = 0$$

These are both second order linear differential equations which can be solved using basic techniques. Their solutions are

$$u_1 = c_{11} \cos\left(\sqrt{\frac{3k}{m}} t\right) + c_{12} \sin\left(\sqrt{\frac{3k}{m}} t\right)$$

$$u_2 = c_{21} \cos\left(\sqrt{\frac{k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{k}{m}} t\right)$$

The solution to the original system of differential equations

$\vec{\ddot{X}} = \frac{k}{m} A \vec{X}$  is found by substituting the values for both the matrix  $P$  and the vector  $\vec{U}$  into the equation  $\vec{X} = P \vec{U}$ .

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_{11} \cos\left(\sqrt{\frac{3k}{m}} t\right) + c_{12} \sin\left(\sqrt{\frac{3k}{m}} t\right) \\ c_{21} \cos\left(\sqrt{\frac{k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{k}{m}} t\right) \end{pmatrix}$$

Multiplying the matrices on the right side together and equating components, we get the following solutions to the differential equation that models the spring-mass system.

$$x_1 = c_{11} \cos\left(\sqrt{\frac{3k}{m}} t\right) + c_{12} \sin\left(\sqrt{\frac{3k}{m}} t\right) + c_{21} \cos\left(\sqrt{\frac{k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{k}{m}} t\right)$$

$$x_2 = -c_{11} \cos\left(\sqrt{\frac{3k}{m}} t\right) - c_{12} \sin\left(\sqrt{\frac{3k}{m}} t\right) + c_{21} \cos\left(\sqrt{\frac{k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{k}{m}} t\right)$$

### Exercise 2.3

Since we have  $n$  variables  $x_1, x_2, \dots, x_n$ , we have  $n$  equations of motion which are

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = -\frac{\partial V}{\partial x_1}, \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = -\frac{\partial V}{\partial x_2}, \quad \dots, \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_n} \right] = -\frac{\partial V}{\partial x_n}$$

The easiest way to construct these equations is to find each component. To find the left side of each of the equations of motion, we first differentiate the equation for kinetic energy with respect to  $\dot{x}_i$  ( $i=1, 2, \dots, n$ ).



$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{2} m [2 \dot{x}_1 + 0 + \dots + 0] = m \dot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2} m [0 + 2 \dot{x}_2 + 0 + \dots + 0] = m \dot{x}_2$$

$$\vdots$$

$$\frac{\partial T}{\partial \dot{x}_n} = \frac{1}{2} m [0 + \dots + 0 + 2 \dot{x}_n] = m \dot{x}_n$$

When we differentiate each of these with respect to time, we have

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = \frac{d}{dt} [m \dot{x}_1] = m \ddot{x}_1$$

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = \frac{d}{dt} [m \dot{x}_2] = m \ddot{x}_2$$

$$\vdots$$

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_n} \right] = \frac{d}{dt} [m \dot{x}_n] = m \ddot{x}_n$$

The right side of each of the equations of motion is

$$-\frac{\partial V}{\partial x_1} = -\frac{1}{2}k[2b_{11}x_1 + 2b_{12}x_2 + \dots + 2b_{1n}x_n] = -k[b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n]$$

$$-\frac{\partial V}{\partial x_2} = -\frac{1}{2}k[2b_{12}x_1 + 2b_{22}x_2 + \dots + 2b_{2n}x_n] = -k[b_{12}x_1 + b_{22}x_2 + \dots + b_{2n}x_n]$$

$$\vdots$$

$$-\frac{\partial V}{\partial x_n} = -\frac{1}{2}k[2b_{1n}x_1 + 2b_{2n}x_2 + \dots + 2b_{nn}x_n] = -k[b_{1n}x_1 + b_{2n}x_2 + \dots + b_{nn}x_n]$$

If we combine these components, the equations of motion become

$$\ddot{x}_1 = \frac{k}{m}[-b_{11}x_1 - b_{12}x_2 - \dots - b_{1n}x_n]$$

$$\ddot{x}_2 = \frac{k}{m}[-b_{12}x_1 - b_{22}x_2 - \dots - b_{2n}x_n]$$

$$\vdots$$

$$\ddot{x}_n = \frac{k}{m}[-b_{1n}x_1 - b_{2n}x_2 - \dots - b_{nn}x_n]$$

The equations of motion can be rewritten in matrix form as

$$\vec{\ddot{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -b_{11} & -b_{12} & \dots & -b_{1n} \\ -b_{12} & -b_{22} & \dots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1n} & -b_{2n} & \dots & -b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{k}{m} \mathbf{B} \vec{X}$$

where  $B$  is an  $n \times n$  symmetric matrix. Now we have an equation that should look very familiar to us.

$$\vec{\ddot{X}} = \frac{k}{m} B \vec{X}$$

Since  $B$  is a symmetric  $n \times n$  matrix, there exists an orthogonal matrix  $P$  such that  $P^{-1}BP = D$ . The matrix  $D$  is the diagonal matrix whose entries along the main diagonal consist of the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $B$  and the columns of  $P$  are the corresponding

eigenvectors  $\vec{X}_{\lambda_1}, \dots, \vec{X}_{\lambda_n}$  associated with the eigenvalues  $\lambda_1, \dots, \lambda_n$ ,

respectively. The following theory will be very similar to the theory that we developed for the spring-mass system with two blocks, except the sizes of the matrices and vectors will be  $n \times n$  and  $n \times 1$ , respectively. If we multiply both sides of this equation by  $P^{-1}$  and use the identity  $PP^{-1} = I_n$ , we get

$$P^{-1}\vec{\ddot{X}} = P^{-1}\frac{k}{m}B\vec{X} = \frac{k}{m}P^{-1}B(P P^{-1})\vec{X} = \frac{k}{m}(P^{-1}BP)(P^{-1}\vec{X})$$

To simplify this equation, we let  $\vec{U} = P^{-1}\vec{X}$ . To introduce the vector variable  $\vec{U}$ , we substitute  $\vec{U} = P^{-1}\vec{X}$  into this equation. We then substitute  $P^{-1}BP = D$  and  $\vec{U} = P^{-1}\vec{X}$  into the right side to obtain

$$\vec{\ddot{U}} = \frac{k}{m}D\vec{U}. \text{ Now, multiplying both sides of this matrix equation by}$$

the matrix  $P$ , we get  $P \vec{U} = \frac{k}{m} P D \vec{U}$ . Rewriting the left side of

$P \vec{U} = \frac{k}{m} P D \vec{U}$  gives

$$P \vec{U} = \begin{pmatrix} P^{(1)} & P^{(2)} & \dots & P^{(n)} \end{pmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_n \end{pmatrix} = P^{(1)} \ddot{u}_1 + P^{(2)} \ddot{u}_2 + \dots + P^{(n)} \ddot{u}_n ,$$

where  $P^{(i)}$  represents the  $i$ th column ( $i = 1, 2, \dots, n$ ) of the matrix  $P$ .

We rewrite the right side to obtain

$$\frac{k}{m} P D \vec{U} = \frac{k}{m} \left[ P^{(1)} \lambda_1 u_1 + P^{(2)} \lambda_2 u_2 + \dots + P^{(n)} \lambda_n u_n \right]$$

Thus,  $P \vec{U} = \frac{k}{m} P D \vec{U}$  can be written as

$$P^{(1)} \ddot{u}_1 + P^{(2)} \ddot{u}_2 + \dots + P^{(n)} \ddot{u}_n = \frac{k}{m} \left[ P^{(1)} \lambda_1 u_1 + P^{(2)} \lambda_2 u_2 + \dots + P^{(n)} \lambda_n u_n \right]$$

We can simplify this equation by multiplying through by  $\frac{k}{m}$ ,

gathering terms, moving everything to the left side, and factoring out  $P^{(1)}, P^{(2)}, \dots, P^{(n)}$  from each quantity.

$$\left[ \ddot{u}_1 - \frac{k}{m} \lambda_1 u_1 \right] P^{(1)} + \left[ \ddot{u}_2 - \frac{k}{m} \lambda_2 u_2 \right] P^{(2)} + \dots + \left[ \ddot{u}_n - \frac{k}{m} \lambda_n u_n \right] P^{(n)} = \vec{0}$$

Since the columns of  $P$  are orthogonal, we know they are linearly independent, thus, the coefficients of the vectors  $P^{(1)}, P^{(2)}, \dots, P^{(n)}$

must be zero. If we set each of the coefficients in the above equation equal to zero, we have

$$\ddot{u}_1 - \frac{k}{m} \lambda_1 u_1 = 0, \quad \ddot{u}_2 - \frac{k}{m} \lambda_2 u_2 = 0, \quad \dots, \quad \ddot{u}_n - \frac{k}{m} \lambda_n u_n = 0.$$

These are all second order linear differential equations which can be solved using basic techniques. If we let  $r_i = -\frac{k}{m} \lambda_i$ , where  $i=1, 2, \dots, n$ , then these equations become

$$\ddot{u}_1 + r_1 u_1 = 0, \quad \ddot{u}_2 + r_2 u_2 = 0, \quad \dots, \quad \ddot{u}_n + r_n u_n = 0.$$

Using the following formulas, we can solve for the vector  $\vec{U}$ . (Note: To determine whether  $r_i$  is zero, negative or positive, substitute  $\lambda_i$

into  $r_i = -\frac{k}{m} \lambda_i$ .)

If  $r_i = 0$ , then  $u_i = c_{i1}t + c_{i2}$

If  $r_i < 0$ , then  $u_i = c_{i1}e^{r_i t} + c_{i2}e^{-r_i t}$

If  $r_i > 0$ , then  $u_i = c_{i1} \cos(\sqrt{r_i} t) + c_{i2} \sin(\sqrt{r_i} t)$ .

The solution to the original system of differential equations

$\ddot{\vec{X}} = \frac{k}{m} \vec{B} \vec{X}$  is found by substituting the values for both the matrix  $\vec{P}$  and the vector  $\vec{U}$  into the equation  $\ddot{\vec{X}} = \vec{P} \vec{U}$ .

#### Exercise 2.4

Since the spring-mass system lies free in the  $xy$ -plane, the entire system can move vertically up or down, horizontally to the left or right, or rotate. These types of motion are called rigid motions. The system can also vibrate producing the motions that are described by Figures 2.6 and 2.7.

#### Exercise 2.5

(a) Suppose the three masses are moved to the right causing the first three springs to stretch by different amounts and causing the fourth spring to be compressed. This is depicted in Figure B.1.

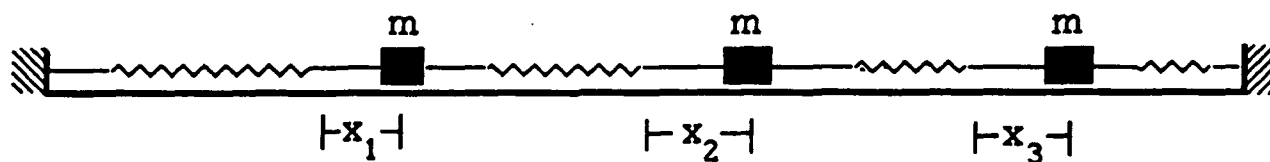


Figure B.1

First, we need the equation which describes the kinetic energy of the system in Figure B.1.

$$T = \frac{1}{2} m [\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2]$$

The potential energy of the system is the sum of the potential energies of each spring. Spring 1 is stretched from its equilibrium position by the amount  $x_1$ , so the potential energy for spring 1 is

$V_1 = \frac{1}{2} k x_1^2$ . Spring 2 is stretched from its equilibrium position by the amount  $x_1 - x_2$ , so that  $V_2 = \frac{1}{2} k (x_1 - x_2)^2$  is the potential energy

for spring 2. Spring 3 is stretched from its equilibrium position by the amount  $x_2 - x_3$ , producing a potential energy of

$V_3 = \frac{1}{2} k (x_2 - x_3)^2$  for spring 3. Spring 4 is compressed from its equilibrium position by the amount  $x_3$ . Thus, the potential energy for spring 4 is  $V_4 = \frac{1}{2} k x_3^2$ . Therefore, the potential energy of the system is

$$\begin{aligned} V &= \sum_{i=1}^4 V_i = \frac{1}{2} k [x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2] \\ &= k [x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3] \end{aligned}$$

Since we have three variables  $x_1$ ,  $x_2$ , and  $x_3$ , we have three equations of motion which are

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = -\frac{\partial V}{\partial x_1}, \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = -\frac{\partial V}{\partial x_2} \quad \text{and} \quad \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_3} \right] = -\frac{\partial V}{\partial x_3}.$$

The easiest way to construct these equations is to find each component. To find the left side of each of the equations of motion, we first differentiate the equation for kinetic energy with respect to  $\dot{x}_i$  ( $i=1,2,3$ ).

$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{2} m [2 \dot{x}_1 + 0 + 0] = m \dot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2} m [0 + 2 \dot{x}_2 + 0] = m \dot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = \frac{1}{2} m [0 + 0 + 2 \dot{x}_3] = m \dot{x}_3$$

Now, differentiating each of these with respect to time, we have



$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_1} \right] = \frac{d}{dt} [m \dot{x}_1] = m \ddot{x}_1$$

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_2} \right] = \frac{d}{dt} [m \dot{x}_2] = m \ddot{x}_2$$

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{x}_3} \right] = \frac{d}{dt} [m \dot{x}_3] = m \ddot{x}_3$$

The right side of each of the equations of motion is

$$-\frac{\partial V}{\partial x_1} = -k[2x_1 - x_2] = k[-2x_1 + x_2]$$

$$-\frac{\partial V}{\partial x_2} = -k[2x_2 - x_1 - x_3] = k[x_1 - 2x_2 + x_3]$$

$$-\frac{\partial V}{\partial x_3} = -k[2x_3 - x_2] = k[x_2 - 2x_3]$$

If we equate these components, the equations of motion become

$$\ddot{x}_1 = \frac{k}{m} [-2x_1 + x_2]$$

$$\ddot{x}_2 = \frac{k}{m} [x_1 - 2x_2 + x_3]$$

$$\ddot{x}_3 = \frac{k}{m} [x_2 - 2x_3]$$

The equations of motion can be rewritten in matrix form, which is the system of differential equations modeling the spring-mass system in Figure B.1.

$$\ddot{\vec{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m} A \ddot{\vec{X}}$$

(b) Since  $A$  is a symmetric matrix, all of its eigenvalues are real and  $A$  is diagonalizable. We begin by finding the eigenvalues of the matrix  $A$ .

$$\det(A - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 1 & 0 \\ 1 & -2 - \lambda & 1 \\ 0 & 1 & -2 - \lambda \end{pmatrix} = (\lambda + 2)(\lambda + 2 - \sqrt{2})(\lambda + 2 + \sqrt{2})$$

If we set  $\det(A - \lambda I)$  equal to zero and solve for  $\lambda$ , we find the eigenvalues are  $\lambda_1 = -2$ ,  $\lambda_2 = -2 + \sqrt{2}$  and  $\lambda_3 = -2 - \sqrt{2}$ . Thus, there exists

an invertible matrix  $P$  such that  $P^{-1}AP = D$ .  $D$  is the diagonal matrix whose entries along the main diagonal consist of the eigenvalues of  $A$  and the columns of  $P$  are the corresponding eigenvectors. To find  $P$ , we need to find the eigenvectors associated with  $\lambda_1 = -2$ ,  $\lambda_2 = -2 + \sqrt{2}$

and  $\lambda_3 = -2 - \sqrt{2}$ . For  $\lambda_1 = -2$  we have to reduce the following

augmented matrix

$$\left( \begin{array}{ccc|c} -2+2 & 1 & 0 & 0 \\ 1 & -2+2 & 1 & 0 \\ 0 & 1 & -2+2 & 0 \end{array} \right)$$

to obtain  $\left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$ , which yields the following equations:  $x_1 = -$

$x_3$  and  $x_2 = 0$ . If we let  $x_1 = -1$ , an eigenvector associated with  $\lambda_1 = -2$

$$\text{is } \vec{X}_{\lambda_1 = -2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For  $\lambda_2 = -2 + \sqrt{2}$  we have to reduce the following augmented matrix.

$$\left( \begin{array}{ccc|c} -2 - (-2 + \sqrt{2}) & 1 & 0 & 0 \\ 1 & -2 - (-2 + \sqrt{2}) & 1 & 0 \\ 0 & 1 & -2 - (-2 + \sqrt{2}) & 0 \end{array} \right),$$

to obtain  $\left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$ , which yields the following equations:

$x_1 = x_3$  and  $x_2 = \sqrt{2}x_3$ . If we let  $x_3 = 1$ , an eigenvector associated with

$$\lambda_2 = -2 + \sqrt{2} \text{ is } \vec{X}_{\lambda_2 = -2 + \sqrt{2}} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

For  $\lambda_2 = -2 - \sqrt{2}$  we have to reduce the following augmented matrix.

$$\left( \begin{array}{ccc|c} -2 - (-2 - \sqrt{2}) & 1 & 0 & 0 \\ 1 & -2 - (-2 - \sqrt{2}) & 1 & 0 \\ 0 & 1 & -2 - (-2 - \sqrt{2}) & 0 \end{array} \right)$$

to obtain  $\left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$ , which yields the following

equations:  $x_1 = x_3$  and  $x_2 = -\sqrt{2}x_3$ . If we let  $x_3 = 1$ , an eigenvector

associated with  $\lambda_1 = -2 - \sqrt{2}$  is  $\vec{x}_{\lambda_1 = -2 - \sqrt{2}} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ . Using the theory

that we developed in Exercise 2.3, we know that we must first solve

the differential equation  $P \vec{U} = \frac{k}{m} P D \vec{U}$ , which leads to

$$\left[ \ddot{u}_1 - \frac{k}{m} \lambda_1 u_1 \right] P^{(1)} + \left[ \ddot{u}_2 - \frac{k}{m} \lambda_2 u_2 \right] P^{(2)} + \left[ \ddot{u}_3 - \frac{k}{m} \lambda_3 u_3 \right] P^{(3)} = \vec{0}$$

Since the columns of  $P$  are orthogonal, we know they are linearly independent. Thus, the coefficients of the vectors  $P^{(1)}$ ,  $P^{(2)}$ , and  $P^{(3)}$ , must be zero. If we set each of the coefficients in the

above equation equal to zero, we have

$$\ddot{u}_1 - \frac{k}{m} \lambda_1 u_1 = 0, \quad \ddot{u}_2 - \frac{k}{m} \lambda_2 u_2 = 0, \quad \ddot{u}_3 - \frac{k}{m} \lambda_3 u_3 = 0$$

Substituting in the eigenvalues, we obtain

$$\ddot{u}_1 - \frac{k}{m}(-2)u_1 = 0, \quad \ddot{u}_2 - \frac{k}{m}(-2 + \sqrt{2})u_2 = 0, \quad \ddot{u}_3 - \frac{k}{m}(-2 - \sqrt{2})u_3 = 0$$

After simplifying these equations they become

$$\ddot{u}_1 + \frac{2k}{m}u_1 = 0, \quad \ddot{u}_2 + \frac{(2 - \sqrt{2})k}{m}u_2 = 0, \quad \ddot{u}_3 + \frac{(2 + \sqrt{2})k}{m}u_3 = 0$$

From these differential equations we observe that

$$r_1 = \frac{2k}{m}, \quad r_2 = \frac{(2 - \sqrt{2})k}{m}, \quad r_3 = \frac{(2 + \sqrt{2})k}{m}$$

Since each  $r_i$  ( $i=1, 2, 3$ ) is greater than zero, the solutions to these second order linear differential equations are

$$u_1 = c_{11} \cos\left(\sqrt{\frac{2k}{m}} t\right) + c_{12} \sin\left(\sqrt{\frac{2k}{m}} t\right)$$

$$u_2 = c_{21} \cos\left(\sqrt{\frac{(2 - \sqrt{2})k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{(2 - \sqrt{2})k}{m}} t\right)$$

$$u_3 = c_{31} \cos\left(\sqrt{\frac{(2 + \sqrt{2})k}{m}} t\right) + c_{32} \sin\left(\sqrt{\frac{(2 + \sqrt{2})k}{m}} t\right)$$

The solution to the original system of differential equations

$\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$  is found by substituting the values for both the matrix  $P$  and the vector  $\vec{U}$  into the equation  $\vec{X} = P \vec{U}$ .

$$\vec{X} = P \vec{U}$$

$$= \begin{pmatrix} -1 & 1 & -\sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_{11} \cos\left(\sqrt{\frac{2k}{m}} t\right) + c_{12} \sin\left(\sqrt{\frac{2k}{m}} t\right) \\ c_{21} \cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}} t\right) \\ c_{31} \cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}} t\right) + c_{32} \sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}} t\right) \end{pmatrix}$$

Multiplying the matrices on the right side together and equating components, we get the following solutions to the differential equation which model the spring-mass system.

$$\begin{aligned} x_1 = & -c_{11} \cos\left(\sqrt{\frac{2k}{m}} t\right) - c_{12} \sin\left(\sqrt{\frac{2k}{m}} t\right) \\ & + c_{21} \cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}} t\right) + c_{22} \sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}} t\right) \\ & + c_{31} \cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}} t\right) + c_{32} \sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}} t\right) \end{aligned}$$

$$x_2 = \sqrt{2} c_{21} \cos \left( \sqrt{\frac{(2-\sqrt{2})k}{m}} t \right) + \sqrt{2} c_{22} \sin \left( \sqrt{\frac{(2-\sqrt{2})k}{m}} t \right) \\ - \sqrt{2} c_{31} \cos \left( \sqrt{\frac{(2+\sqrt{2})k}{m}} t \right) - \sqrt{2} c_{32} \sin \left( \sqrt{\frac{(2+\sqrt{2})k}{m}} t \right)$$

$$x_3 = c_{11} \cos \left( \sqrt{\frac{2k}{m}} t \right) + c_{12} \sin \left( \sqrt{\frac{2k}{m}} t \right) \\ + c_{21} \cos \left( \sqrt{\frac{(2-\sqrt{2})k}{m}} t \right) + c_{22} \sin \left( \sqrt{\frac{(2-\sqrt{2})k}{m}} t \right) \\ + c_{31} \cos \left( \sqrt{\frac{(2+\sqrt{2})k}{m}} t \right) + c_{32} \sin \left( \sqrt{\frac{(2+\sqrt{2})k}{m}} t \right)$$

(c) To describe the configurations in which the spring-mass system vibrates, we need to write each  $u_i$  in terms of the  $x_i$ . This can be

done by using the matrix equation  $\vec{X} = P \vec{U}$  or  $\vec{U} = P^{-1} \vec{X}$ . We first find  $P^{-1}$  and substitute it into the matrix equation.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Multiplying these matrices together and equating components, we get the following three equations.

$$u_1 = \frac{-x_1 + x_3}{2}$$

$$u_2 = \frac{x_1 + \sqrt{2} x_2 + x_3}{4}$$

$$u_3 = \frac{x_1 - \sqrt{2} x_2 + x_3}{4}$$

These three equations describe the relationships between the variables

$x_1$ ,  $x_2$  and  $x_3$ . The three modes in which this spring-mass system

vibrates are  $u_1 = \frac{-x_1 + x_3}{2}$  with  $u_2=0$  and  $u_3=0$ ,

$u_2 = \frac{x_1 + \sqrt{2} x_2 + x_3}{4}$  with  $u_1=0$  and  $u_3=0$ , and

$u_3 = \frac{x_1 - \sqrt{2} x_2 + x_3}{4}$  with  $u_1=0$  and  $u_2=0$ .

In the first mode,  $u_1 = \frac{-x_1 + x_3}{2}$  with  $u_2=0$  and  $u_3=0$ . Since

$\frac{-x_1 + x_3}{2}$  represents how the distance between blocks one and three is changing, the first mode of vibration describes how the



distance between the two blocks is changing. We visualize this by considering a series of diagrams similar to those in Figure 2.6.

Recall, the banner is made of an elastic material and indicates the distance between the two blocks. When this system vibrates, we see the banner contracting and stretching with a frequency associated with  $\lambda_1$ . This is indicated by the series of "snapshots" of the spring-mass system in motion seen in Figure B.2.

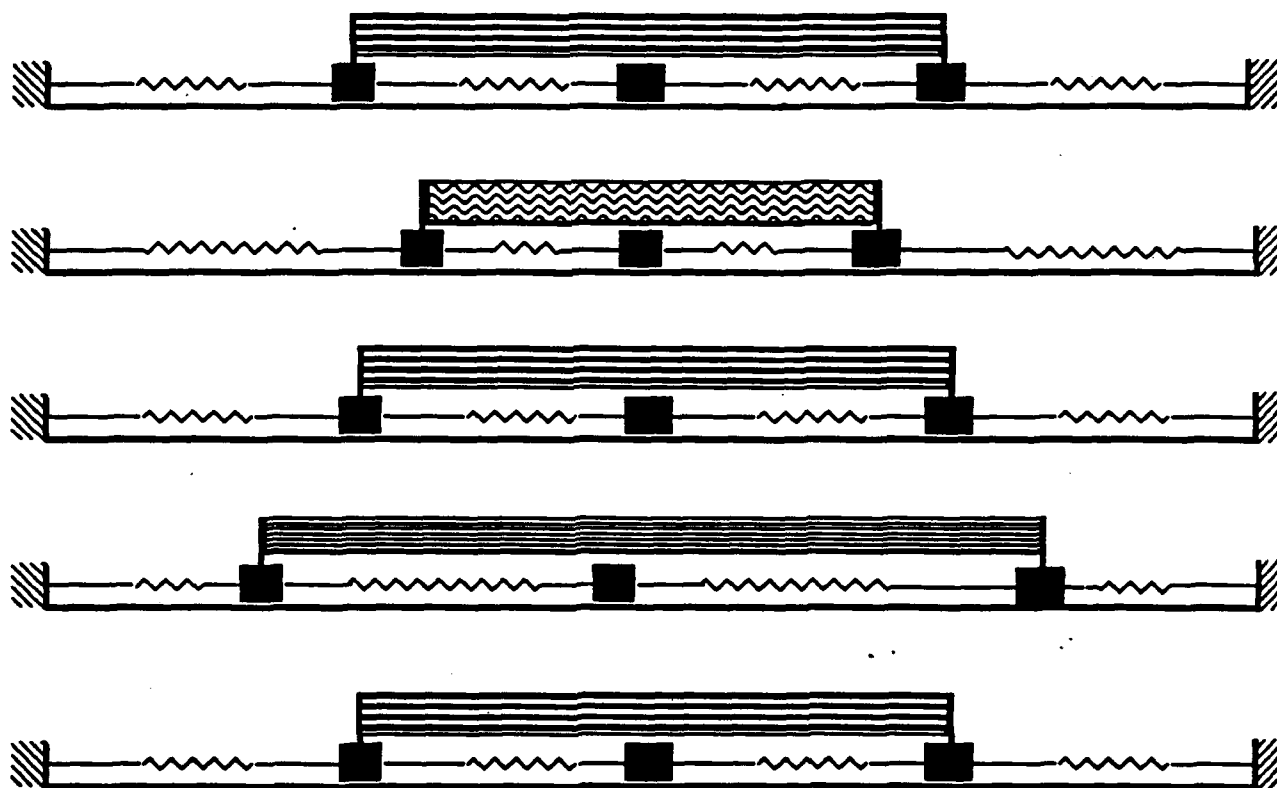


Figure B.2

In the second mode,  $u_2 = \frac{x_1 + \sqrt{2} x_2 + x_3}{4}$  with  $u_1 = 0$  and

$u_3 = 0$ . To visualize this mode of vibration, we consider a series of diagrams similar to those in Figure 2.6. When this system vibrates, we see the flag moving back and forth with a frequency associated with  $\lambda_2$ . This is indicated by the series of "snapshots" of the spring-mass system in motion as seen in Figure B.3.

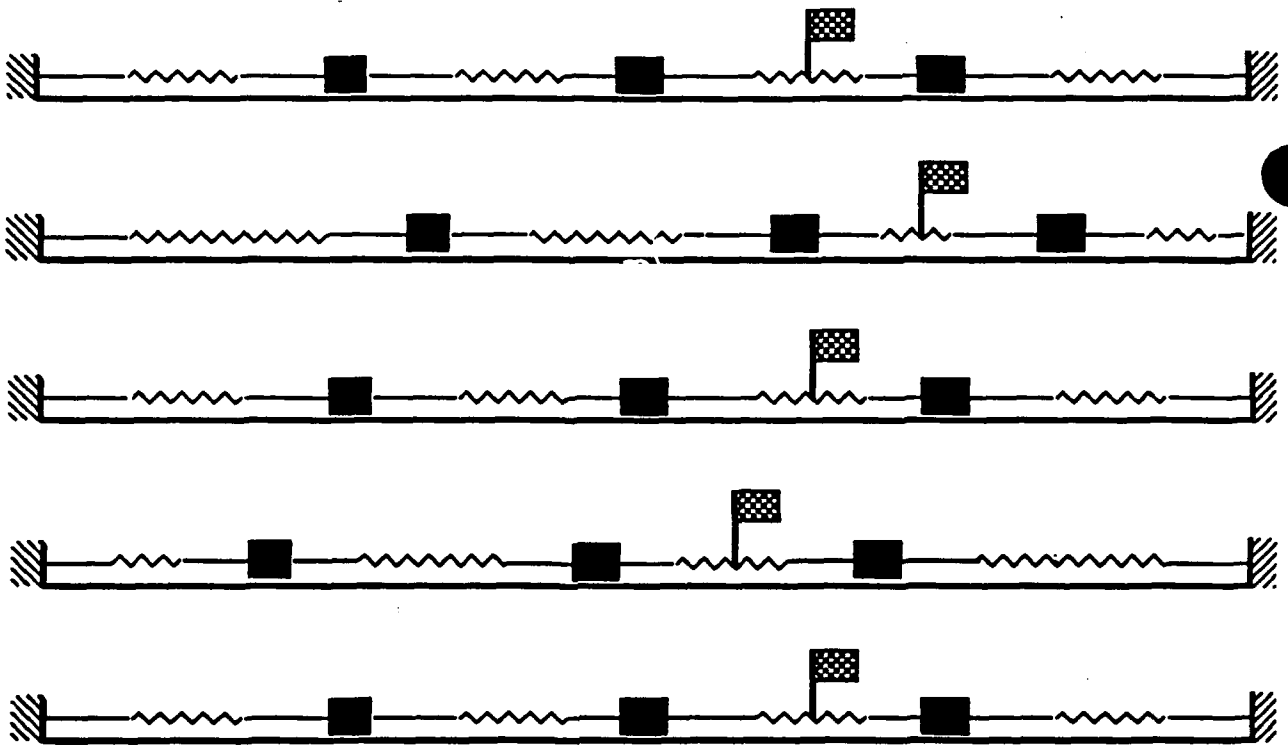


Figure B.3

In the third mode,  $u_3 = \frac{x_1 - \sqrt{2} x_2 + x_3}{4}$  with  $u_1=0$  and  $u_2=0$ .

To visualize this mode of vibration, we consider a series of diagrams similar to those in Figure 2.6. When this system vibrates, we see the flag moving back and forth with a frequency associated with  $\lambda_3$ .

This is indicated by the series of "snapshots" of the spring-mass system in motion, as seen in Figure B.4.

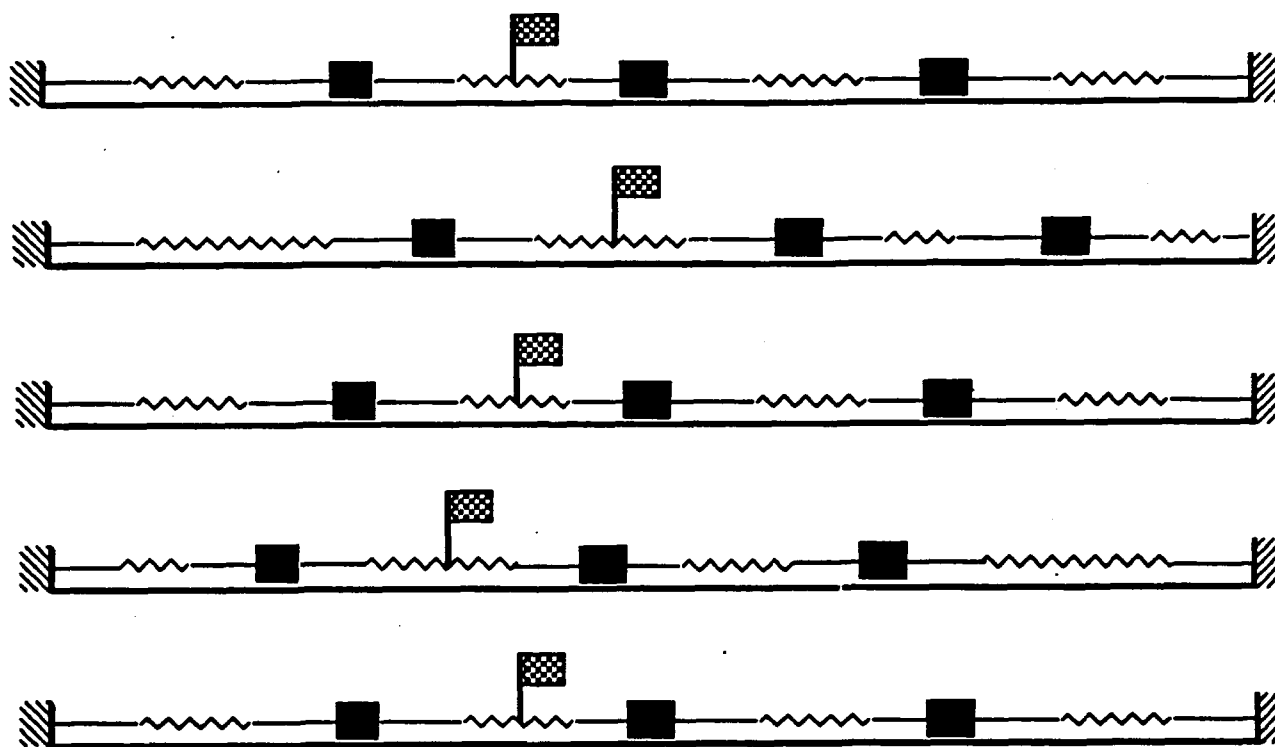


Figure B.4

Exercise 2.6

Since the spring-mass system lies in the  $xy$ -plane, the entire system can move vertically up or down, horizontally to the left or right, or rotate. Besides these rigid motions, the system can also vibrate producing the motion that were described in Exercise 2.6.

Exercise 3.1

We write  $f\left(\frac{1}{L}\right)$  as a Taylor series expansion expanded about zero.

$$f\left(\frac{1}{L}\right) = f(0) + \frac{f'(0)}{1!} \frac{1}{L} + \frac{f''(0)}{2!} \left(\frac{1}{L}\right)^2 + \dots$$

We use  $f\left(\frac{1}{L}\right)$ , which has been repeated below for convenience, to find  $f(0)$ ,  $f'(0)$  and  $f''(0)$ .

$$f\left(\frac{1}{L}\right) = \left\{ 1 + \frac{1}{L} \left( (x_1 - x_2) - \sqrt{3} (y_1 - y_2) \right) + \frac{1}{L^2} \left( (x_1 - x_2)^2 + (y_1 - y_2)^2 \right) \right\}^{\frac{1}{2}}$$

$$f(0) = 1$$

$$f'(0) = \frac{1}{2} \left[ (x_1 - x_2) - \sqrt{3} (y_1 - y_2) \right]$$

$$f''(0) = -\frac{1}{4} \left[ (x_1 - x_2) - (y_1 - y_2) \right]^2 + (x_1 - x_2)^2 - (y_1 - y_2)^2$$

Substituting these into the Taylor series expansion and simplifying,

we have

$$\begin{aligned}
 f\left(\frac{1}{L}\right) &= 1 + \frac{\frac{1}{2}(x_1 - x_2) - \sqrt{3}(y_1 - y_2)}{1!} \frac{1}{L} \\
 &\quad + \frac{-\frac{1}{4}[(x_1 - x_2) - (y_1 - y_2)]^2 + (x_1 - x_2)^2 - (y_1 - y_2)^2}{2!} \frac{1}{L^2} + \dots \\
 &= 1 + \frac{1}{2}(x_1 - x_2) - \sqrt{3}(y_1 - y_2) \frac{1}{L} + \text{higher power terms.}
 \end{aligned}$$

### Exercise 3.2

Figure B.5 contains the coordinates of the blocks and the distance formula for the spring from Figure 3.3.

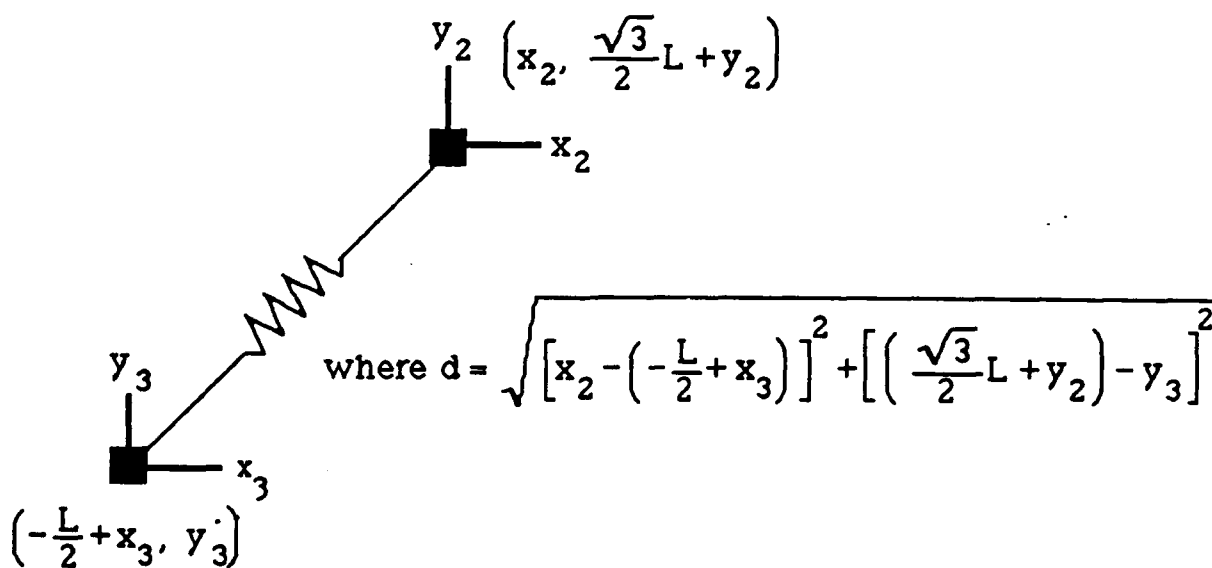


Figure B.5

The potential energy of this spring is one half the spring constant  $k$  times the square of the distance that the spring is stretched. The displacement of the spring from its equilibrium position (the distance that the spring is stretched) is  $|d-L|$ . Expressing the potential energy of the spring in terms of  $|d-L|$ , we have

$$V_{23} = \frac{1}{2} k |d-L|^2$$

We want to rewrite  $V_{23}$  using the variables  $x_2$ ,  $y_2$ ,  $x_3$  and  $y_3$ . To do this, we must first simplify the expression for the distance  $d$ .

$$\begin{aligned} d &= \left\{ \left[ x_2 - \left( -\frac{L}{2} + x_3 \right) \right]^2 + \left[ \left( \frac{\sqrt{3}}{2} L + y_2 \right) - y_3 \right]^2 \right\}^{\frac{1}{2}} \\ &= \left\{ \left[ \frac{L}{2} + (x_2 - x_3) \right]^2 + \left[ \frac{\sqrt{3}}{2} L + (y_2 - y_3) \right]^2 \right\}^{\frac{1}{2}} \\ &= L \left\{ 1 + \frac{1}{L} \left[ (x_2 - x_3) + \sqrt{3} (y_2 - y_3) \right] + \frac{1}{L^2} \left[ (x_2 - x_3)^2 + (y_2 - y_3)^2 \right] \right\}^{\frac{1}{2}} \end{aligned}$$

We can rewrite the quantity expressed by the square root using its Taylor series expansion. The terms of higher powers have been grouped together for convenience.

$$d = L \left\{ 1 + \frac{\frac{1}{2} \left[ (x_2 - x_3) + \sqrt{3} (y_2 - y_3) \right]}{1!} \frac{1}{L} + \text{terms of higher powers} \right\}$$

This expression can be simplified by multiplying through by  $L$  and then moving the  $L$  to the left side. The resulting quantity is what we wanted to find.

$$d - L = \frac{1}{2} \left[ (x_2 - x_3) + \sqrt{3} (y_2 - y_3) \right] + \text{terms of higher powers}$$

This quantity can now be substituted into the formula for the potential energy  $V_{23}$ .

$$V_{23} = \frac{1}{2} k |d - L|^2$$

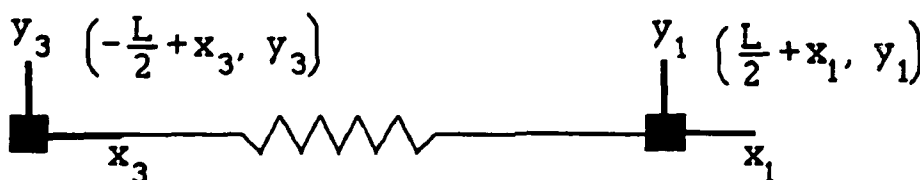
$$= \frac{k}{2} \left\{ \frac{1}{4} \left[ (x_2 - x_3) + \sqrt{3} (y_2 - y_3) \right]^2 + \text{terms of higher powers} \right\}$$

The Taylor series expansion for potential energy can not have nonzero linear terms, since the system has an equilibrium configuration. Also, we are considering only small vibrations so we ignore the terms of higher powers. The formula for potential energy  $V_{23}$ , is

$$\begin{aligned}
 v_{23} &= \frac{k}{2} \left\{ \frac{1}{4} \left[ (x_2 - x_3) + \sqrt{3} (y_2 - y_3) \right]^2 \right\} \\
 &= \frac{k}{2} \left\{ \frac{x_2^2}{4} + \frac{x_3^2}{4} + \frac{3y_2^2}{4} + \frac{3y_3^2}{4} - \frac{1}{2} x_2 x_3 + \frac{\sqrt{3}}{2} x_2 y_2 - \frac{\sqrt{3}}{2} x_3 y_2 \right. \\
 &\quad \left. - \frac{\sqrt{3}}{2} x_2 y_3 + \frac{\sqrt{3}}{2} x_3 y_3 - \frac{3}{2} y_2 y_3 \right\}
 \end{aligned}$$

### Exercise 3.3

Figure B.6 contains the coordinates of the blocks and the distance formula for the spring from Figure 3.4.



$$\text{where } d = \sqrt{\left[ \left( \frac{L}{2} + x_1 \right) - \left( -\frac{L}{2} + x_3 \right) \right]^2 + [y_1 - y_3]^2}$$

Figure B.6

The potential energy of this spring is one half the spring constant  $k$ , times the square of the distance that the spring is stretched. The displacement of the spring from its equilibrium position (the distance that the spring is stretched) is  $|d - L|$ . Expressing the potential energy of the spring in terms of  $|d - L|$ , we have



$$V_{13} = \frac{1}{2} k |d - L|^2$$

We want to rewrite  $V_{13}$  using the variables  $x_1$ ,  $y_1$ ,  $x_3$  and  $y_3$ . To do this we must first simplify the expression for the distance  $d$ .

$$\begin{aligned} d &= \left\{ \left[ \left( \frac{L}{2} + x_1 \right) - \left( -\frac{L}{2} + x_3 \right) \right]^2 + \left[ y_1 - y_3 \right]^2 \right\}^{\frac{1}{2}} \\ &= \left\{ L^2 + 2L(x_1 - x_3) + (x_1 - x_3)^2 + (y_1 - y_3)^2 \right\}^{\frac{1}{2}} \\ &= L \left\{ 1 + \frac{2}{L}(x_1 - x_3) + \frac{1}{L^2} \left[ (x_1 - x_3)^2 + (y_1 - y_3)^2 \right] \right\}^{\frac{1}{2}} \end{aligned}$$

We can rewrite the quantity expressed by the square root using its Taylor series expansion. The terms of higher powers have been grouped together for convenience.

$$d = L \left\{ 1 + \frac{\frac{1}{2} \left[ 2(x_1 - x_3) \right]}{1!} \frac{1}{L} + \text{higher power terms} \right\}$$

This expression can be simplified by multiplying through by  $L$  and then moving the  $L$  to the left side. The resulting quantity is what we wanted to find.

$$d - L = (x_1 - x_3) + \text{higher power terms}$$

This quantity can now be substituted into the formula for the potential energy  $V_{13}$ .

$$\begin{aligned} V_{13} &= \frac{1}{2} k |d - L|^2 \\ &= \frac{k}{2} \left\{ (x_1 - x_3)^2 + \text{higher power terms} \right\} \end{aligned}$$

The Taylor series expansion for potential energy can not have any nonzero linear terms, since the system has an equilibrium configuration. Also, we are considering only small vibrations so we ignore the terms of higher powers. The formula for potential energy  $V_{13}$ , is

$$\begin{aligned} V_{13} &= \frac{k}{2} (x_1 - x_3)^2 \\ &= \frac{k}{2} \left\{ x_1^2 + x_3^2 - 2 x_1 x_3 \right\} \end{aligned}$$

Exercise 3.4

Substituting  $\vec{X}_{\lambda_1=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  into  $V(\vec{X}_{\lambda_j=0}) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$ ,

gives

$$V \left[ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] = (1 \ 1 \ 1 \ 0 \ 0 \ 0) \frac{-k}{8m} A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{-k}{8m} (1 \ 1 \ 1 \ 0 \ 0 \ 0) \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{-k}{8m} (0 \ 0 \ 0 \ 0 \ 0 \ 0) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Exercise 3.5

Substituting  $\vec{X}_{\lambda_2=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$  into  $V(\vec{X}_{\lambda_j=0}) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$ ,

gives

$$V \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right] = (0 \ 0 \ 0 \ 1 \ 1 \ 1) \frac{-k}{8m} A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{-k}{8m} (0 \ 0 \ 0 \ 1 \ 1 \ 1) \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{-k}{8m} (0 \ 0 \ 0 \ 0 \ 0 \ 0) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

Exercise 3.6

Substituting  $\vec{X}_{\lambda_3=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$  into

$$V(\vec{X}_{\lambda_j=0}) = \vec{X}_{\lambda_j=0}^T \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0, \text{ gives}$$

$$V \left[ \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} \right] = (-\sqrt{3}, 2\sqrt{3}, -\sqrt{3}, -3, 0, 3) \frac{-k}{8m} A \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$$

$$= \frac{-k}{8m} (-\sqrt{3}, 2\sqrt{3}, -\sqrt{3}, -3, 0, 3) \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \vec{X}_{\lambda}$$

$$= \frac{-k}{8m} (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} = 0$$

Now, show that  $\vec{X}_{\lambda_3=0}$  is orthogonal to both  $\vec{X}_{\lambda_1=0}$  and  $\vec{X}_{\lambda_2=0}$ .

$$\vec{X}_{\lambda_3=0} \cdot \vec{X}_{\lambda_1=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\vec{X}_{\lambda_3=0} \cdot \vec{X}_{\lambda_2=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

Exercise 3.7

To show that  $\vec{X}_{\lambda_1=-6}$  and  $\vec{X}_{\lambda_2=-6}$  are orthogonal, we show their dot product is zero.

$$\vec{X}_{\lambda_1=-6} \cdot \vec{X}_{\lambda_2=-6} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ -1 \end{pmatrix} = 0$$